

# A Translation of Weighted LTL Formulas to Weighted Büchi Automata over $\omega$ -valuation Monoids

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## Abstract

In this paper we introduce a weighted LTL over product  $\omega$ -valuation monoids that satisfy specific properties. We also introduce weighted generalized Büchi automata with  $\varepsilon$ -transitions, as well as weighted Büchi automata with  $\varepsilon$ -transitions over product  $\omega$ -valuation monoids and prove that these two models are expressively equivalent and also equivalent to weighted Büchi automata already introduced in the literature. We prove that every formula of a syntactic fragment of our logic can be effectively translated to a weighted generalized Büchi automaton with  $\varepsilon$ -transitions. For generalized product  $\omega$ -valuation monoids that satisfy specific properties we define a weighted LTL, weighted generalized Büchi automata with  $\varepsilon$ -transitions, and weighted Büchi automata with  $\varepsilon$ -transitions, and we prove the aforementioned results for generalized product  $\omega$ -valuation monoids as well. The translation of weighted LTL formulas to weighted generalized Büchi automata with  $\varepsilon$ -transitions is now obtained for a restricted syntactical fragment of the logic.

**Keywords:** *weighted automata, valuation monoids, weighted LTL*

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## 1 Introduction

Weighted automata over finite and infinite words, defined in [26] and [15, 16] respectively, are essential models in theoretical computer science suitable to describe quantitative features of systems' behavior. They can be seen as classical automata whose transitions are equipped with some value, usually taken from a semiring. Weighted automata have already been successfully used in applications in digital image compression and natural language processing (cf. Chapters 11 and 14 respectively in [11]), and there is a constantly increasing interest for possible use of these models in other fields also, e.g., in medicine, biology (cf. [31, 32]). Chatterjee, Doyen, and Henzinger in [5] defined automata with weights over the real numbers. The behavior of these automata is not computed with the use of the structure of the semiring. More precisely, the weight of a run (finite or infinite) is computed by using a function that assigns a real value to the (finite or infinite) run of the automaton. Examples of such functions are *Max* and *Sum* for finite runs, and *Sup*, *Limsup*, *Liminf*, limit average, and discounted sum for infinite runs. The real value that eventually the automaton assigns to a word is computed as the maximum (resp. supremum for infinite words) of the values of all possible runs of the automaton on the word. In that work, Chatterjee, Doyen and Henzinger presented answers to decidability problems and studied their computational complexity, and further compared the expressive power of their model for different functions. Similar questions were answered in [6, 8, 7] where other kinds of automata that use functions for the computation of the weight of a run were presented. With the functions mentioned above we can model a wide spectrum of procedures of the behavior of several systems. The peak of power consumption can be modeled as the maximum of a sequence of real numbers that represent power consumption, while average response time can be modeled as the limit average [3, 4]. For a detailed reference on the importance of valuation functions we refer to [5]. Droste and Gastin introduced a weighted MSO logic in [10], and Droste and Meinecke extended this logic in [13] to a weighted MSO logic capable of describing properties of the automata of [5], and introduced the structures of valuation monoids and  $\omega$ -valuation monoids as a formalism capable of describing in a generic way their behavior for different functions. The authors further defined the structures of product valuation monoids and product  $\omega$ -valuation monoids by equipping valuation and  $\omega$ -valuation monoids with a multiplicative operation that is not necessarily associative or commutative. Under the consideration of specific properties

of the aforementioned structures the authors proved for finite (resp. infinite) words the expressive equivalence of syntactical fragments of their logic with weighted automata (resp. weighted Muller automata) whose behavior is computed with the use of valuation functions (resp.  $\omega$ -valuation functions). In [24], the structure of valuation monoids was equipped with a family of product operations, as well as with a Cauchy product and iteration of series, and the expressive equivalence of weighted automata over valuation monoids and weighted rational expressions was proved. In the same work, similar results were obtained for the case of infinite words.

In the field of quantitative description of systems, the interest is also focused in the development of tools able to perform quantitative analysis and verification of systems [17, 25]. A possible road to follow is the definition of quantitative specification languages and the investigation of their relation with weighted automata. Such a study would set the foundations for a successful generalization of the automata theoretic-approach in model-checking (cf. [27, 28]) in the quantitative setup.

An automata theoretic approach for reasoning about multivalued objects was proposed in [19]. More, precisely, the authors defined a weighted *LTL* and weighted automata over De Morgan Algebras and presented a translation of the formulas of the logic to weighted automata. In [20] the author defined a weighted *LTL* with weights and discounting parameters over the max-plus semiring and introduced the model of weighted generalized Büchi automata with  $\varepsilon$ -transitions and discounting. In that work, formulas of a syntactic fragment of the proposed logic were effectively translated to weighted generalized Büchi automata with discounting and  $\varepsilon$ -transitions and this model was proved expressively equivalent to weighted Büchi automata with discounting introduced in [12]. In [21] (Chapter 4) it was shown that the aforementioned translation is also possible for formulas of a larger fragment of that logic.

It is the aim of this work to introduce a weighted *LTL* over product  $\omega$ -valuation monoids capable of describing how the quantitative behavior of systems changes over time and present a translation of formulas of a fragment of the logic to weighted generalized Büchi automata with  $\varepsilon$ -transitions, and provide in this way a theoretical basis for the definition of algorithms that can be used for the verification of quantitative properties of systems. As mentioned before the structure of product  $\omega$ -valuation monoids refers to a wide range of applications.

More precisely, we introduce a weighted *LTL* with weights over product

$\omega$ -valuation monoids (resp. generalized product  $\omega$ -valuation monoids), and prove the results of [21] (Chapter 4) for a restricted syntactical fragment of the proposed logic. In more detail, the content of this paper can be described as follows. After presenting some preliminary notions in Section 3, in Section 4 we present the structures of product  $\omega$ -valuation monoids and generalized product  $\omega$ -valuation monoids and we study their properties. In Section 5, for product  $\omega$ -valuation monoids (resp. generalized product  $\omega$ -valuation monoids) that satisfy specific properties we define the models of weighted generalized Büchi automata with  $\varepsilon$ -transitions and weighted Büchi automata with  $\varepsilon$ -transitions. We prove that these two models are equivalent and also equivalent to weighted Büchi automata over product  $\omega$ -valuation monoids (resp. over generalized product  $\omega$ -valuation monoids). In Section 6, we introduce the weighted *LTL* over product  $\omega$ -valuation monoids that satisfy specific properties and prove that the formulas of a syntactic fragment of the proposed *LTL* can be effectively translated to weighed generalized Büchi automata with  $\varepsilon$ -transitions following the constructive approach of [20]. In Section 7 we obtain the results of Section 6 for a restricted syntactical fragment of the weighted *LTL* over generalized product  $\omega$ -valuation monoids.

## 2 Related Work

We recall from the introduction that weighted versions of *LTL* and translations of formulas of the proposed logics to weighted automata were presented in [19] and in [20] (see also in [21]). Both constructions, the one in [19] and the one in [20] aim to simulate the inductive computation of the semantics of the formulas of the proposed logics, nevertheless different algebraic properties of the underlying structures lead to different constructive approaches. More precisely, in [19] the authors treat the formulas as classical ones where elements of De Morgan Algebras are considered as atomic propositions and obtain by [28] the corresponding Büchi automaton. Then, the automaton is transformed into a weighted one where the weights of the transitions are indicated by the sets of atomic propositions with which the unweighted automaton moves between two states. In [20] the approach of [28] is also followed in the sense that the states of the automaton are sets of formulas satisfying discrete conditions of consistency, and the final subsets are defined with respect to the until operators. Nevertheless, the effective simulation of the computation of the semantics of the given formula requires the existence of a maximal formula (according to subformula relation) in each

state that will indicate the induction (and thus the operations) connecting the formulas  $k \in K$  of the state. In addition, as in [9],  $\varepsilon$ -transitions are used to reduce formulas. However, in [9] the goal of the reduction is the production of sets of formulas whose elements are atomic propositions, or their negations, or formulas with outermost connective the next operator. In the case of [20]  $\varepsilon$ -transitions are used to reduce the maximal formula of a set, and to ensure that the state set of the automaton is finite. In this work we follow the constructive approach of [20], however the lack of algebraic properties, with which every semiring is equipped with, imposes the need for a stronger syntactical restriction on the formulas of our logic, in order to achieve the desired translation of formulas to weighted generalized Büchi automata with  $\varepsilon$ -transitions. Another quantitative version of *LTL* with values over  $[0, 1]$  and discounting parameters is presented in [2], where the authors show that threshold model checking problems can be decided by translating the weighted *LTL* formulas of that logic into Boolean nondeterministic Büchi automata.

Lately, classical results for *LTL* have been generalized in the weighted set-up. More precisely, in [14] the authors proved for (infinitary) series over arbitrary bounded lattices the coincidence of *LTL*-definability, *FO*-definability, star-freeness and aperiodicity. In [22] (cf. also Chapter 5 in [21]) the expressive equivalence of (fragments of) *LTL*-definable, *FO*-definable, star-free and counter-free series infinitary series over the max-plus semiring with discounting was proved. This result was generalized in [23] (cf. also Chapter 5 in [21]) for infinitary series over totally commutative complete, idempotent and zero-divisor free semirings.

### 3 Preliminaries

Let  $C, K$  be sets. If  $B$  is a subset of  $C$  (resp. proper subset of  $C$ ), we shall write  $B \subseteq C$  (resp.  $B \subset C$ ). We shall denote by  $\mathcal{P}(C)$  the powerset of  $C$ . An index set  $I$  of  $C$  is a subset of  $C$  whose elements are used to label the elements of another set. A family of elements of  $K$  over the index set  $I$ , denoted by  $(k_i)_{i \in I}$ , is a mapping  $f$  from  $I$  to  $K$  where  $k_i = f(i)$  for all  $i \in I$ . We shall denote by  $\mathbb{N}$  the set of non-negative integers.

*Words* Let  $A$  be an alphabet, i.e., a finite non-empty set. As usually, we denote by  $A^*$  the set of all finite words over  $A$  and  $A^+ = A^* \setminus \{\varepsilon\}$ , where  $\varepsilon$  is the empty word. The set of all infinite sequences with elements in  $A$ , i.e., the set of all infinite words over  $A$ , is denoted by  $A^\omega$ . Let  $w \in A^\omega$ . A word

$v \in A^\omega$  is called a suffix of  $w$ , if  $w = uv$  for some  $u \in A^*$ . Every infinite word  $w = a_0a_1\dots$  with  $a_i \in A$  ( $i \geq 0$ ) is written also as  $w = w(0)w(1)\dots$  where  $w(i) = a_i$  ( $i \geq 0$ ). The word  $w_{\geq i}$  denotes the suffix of  $w$  that starts at position  $i$ , i.e.,  $w_{\geq i} = w(i)w(i+1)\dots$ .

*Monoids* A monoid  $(K, +, \mathbf{0})$  is an algebraic structure equipped with a non-empty set  $K$  and an associative additive operation  $+$  with a zero element  $\mathbf{0}$ , i.e.,  $\mathbf{0} + k = k + \mathbf{0} = k$  for every  $k \in K$ . The monoid  $K$  is called commutative if  $+$  is commutative.

A monoid  $(K, +, \mathbf{0})$  is called complete if it is equipped, for every index set  $I$ , with an infinitary sum operation  $\sum_I : K^I \rightarrow K$  such that for every family  $(k_i)_{i \in I}$  of elements of  $K$  we have

$$\sum_{i \in \emptyset} k_i = \mathbf{0}, \quad \sum_{i \in \{j\}} k_i = k_j, \quad \sum_{i \in \{j,l\}} k_i = k_j + k_l \text{ for } j \neq l$$

and

$$\sum_{j \in J} \left( \sum_{i \in I_j} k_i \right) = \sum_{i \in I} k_i,$$

if  $\bigcup_{j \in J} I_j = I$  and  $I_j \cap I_{j'} = \emptyset$  for  $j \neq j'$ . We note that every complete monoid is commutative.

Let  $K$  be a complete monoid.  $K$  is called additively idempotent (or simply idempotent), if  $k + k = k$  for every  $k \in K$ . Furthermore,  $K$  is zero-sum free if  $k + k' = \mathbf{0}$  implies  $k = k' = \mathbf{0}$ . It is well known that if  $K$  is idempotent, then  $K$  is necessarily zero-sum free ([1]). We recall (cf. [18]) that idempotency gives rise to a natural partial order in  $K$  defined in the following way. Let  $k, k' \in K$ , then  $k \leq k'$  iff  $k' = k' + k$ . Equivalently, it holds  $k \leq k'$  iff  $k' = k'' + k$  for some  $k'' \in K$  (cf. Chapter 5 in [11]). We recall that a partial order of a set  $K$  is a total order, if  $k \leq k'$ , or  $k' \leq k$  for all  $k, k' \in K$ . Let now  $K', K''$  be two non-empty subsets of a complete monoid  $K$ . We define the sum of  $K'$  and  $K''$  in the following way

$$K' + K'' = \{k \in K \mid \exists k' \in K', k'' \in K'' \text{ s.t. } k = k' + k''\}.$$

*Series* Let  $A$  be an alphabet and  $K$  be a complete monoid. An infinitary series over  $A$  and  $K$  is a mapping  $s : A^\omega \rightarrow K$ . For every  $w \in A^\omega$  we write  $(s, w)$  for the value  $s(w)$  and refer to it as the coefficient of  $s$  on  $w$ . We denote by  $K \langle\langle A^\omega \rangle\rangle$  the class of all infinitary series over  $A^\omega$  and  $K$ .

## 4 Product $\omega$ -valuation Monoids, Generalized Product $\omega$ -valuation Monoids, and their Properties

For a set  $K$  we denote by  $L \subseteq_{fin} K$  the fact that  $L$  is a finite subset of  $K$  and we let  $(K_{fin})^\omega = \bigcup_{L \subseteq_{fin} K} L^\omega$ . We now recall the definition of  $\omega$ -valuation

monoids and product  $\omega$ -valuation monoids from [13], with the difference that we equip these structures with two additional properties.

**Definition 1** *An  $\omega$ -valuation monoid  $(K, +, Val^\omega, \mathbf{0})$  is a complete monoid  $(K, +, \mathbf{0})$  equipped with an  $\omega$ -valuation function  $Val^\omega : (K_{fin})^\omega \rightarrow K$  such that  $Val^\omega(k_i)_{i \in \mathbb{N}} = \mathbf{0}$  whenever  $k_i = \mathbf{0}$  for some  $i \geq 0$ . A product  $\omega$ -valuation monoid  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  is an  $\omega$ -valuation monoid  $(K, +, Val^\omega, \mathbf{0})$  further equipped with a product operation  $\cdot : K^2 \rightarrow K$ , with  $\mathbf{1} \in K, \mathbf{1} \neq \mathbf{0}$ , such that  $Val^\omega(\mathbf{1}^\omega) = \mathbf{1}$  and  $\mathbf{0} \cdot k = k \cdot \mathbf{0} = \mathbf{0}$ ,  $\mathbf{1} \cdot k = k \cdot \mathbf{1} = k$  for all  $k \in K$ ; additionally, for every index set  $I$  and  $k \in K$ ,  $\sum_I (k \cdot \mathbf{1}) = k \cdot \sum_I \mathbf{1}$ , and for every  $L \subseteq_{fin} K$ , finite index sets  $I_j (j \geq 0)$ , and all  $k_{i_j} \in L (i_j \in I_j)$*

$$Val^\omega \left( \sum_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}} = \sum_{(i_j)_{j \in \mathbb{N}} \in I_0 \times I_1 \times \dots} Val^\omega(k_{i_j})_{j \in \mathbb{N}}. \quad (1)$$

The property described by equation 1 expresses the **distributivity** of  $Val^\omega$  over finite sums. We recall that this property has also been considered in [24] for the definition of Cauchy  $\omega$ -indexed valuation monoids.

**Remark 1** *Observe that for every  $k \in K$ , it holds  $k_1 \cdot (k_2 \cdot k_3) = (k_1 \cdot k_2) \cdot k_3$  for every  $k_1, k_2, k_3 \in \{\mathbf{0}, \mathbf{1}, k\}$  such that  $k_i \in K \setminus \{\mathbf{0}, \mathbf{1}\}$  for at most one  $i \in \{1, 2, 3\}$ .*

We introduce now the notion of *generalized product  $\omega$ -valuation monoids*. These are defined with the same way as product  $\omega$ -valuation monoids with the difference that these structures are equipped with a restricted version of the distributivity property of  $Val^\omega$  over finite sums.

**Definition 2** *A generalized product  $\omega$ -valuation monoid  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  is an  $\omega$ -valuation monoid  $(K, +, Val^\omega, \mathbf{0})$  further equipped with a product operation  $\cdot : K^2 \rightarrow K$ , with  $\mathbf{1} \in K, \mathbf{1} \neq \mathbf{0}$ , such that  $Val^\omega(\mathbf{1}^\omega) = \mathbf{1}$  and  $\mathbf{0} \cdot k = k \cdot \mathbf{0} = \mathbf{0}$ ,  $\mathbf{1} \cdot k = k \cdot \mathbf{1} = k$  for all  $k \in K$ ; additionally, for every*

index set  $I$  and  $k \in K$ ,  $\sum_I (k \cdot \mathbf{1}) = k \cdot \sum_I \mathbf{1}$ , and the following hold: Let  $L \subseteq_{fin} K$ , and  $I_j (j \geq 0)$  a family of finite index sets. If for all but a finite number of  $j \geq 0$ , one of the following holds  $k_{i_j} \in L \setminus \{\mathbf{0}, \mathbf{1}\}$  for all  $i_j \in I_j$ , or  $k_{i_j} \in \{\mathbf{0}, \mathbf{1}\}$  for all  $i_j \in I_j$ , we have

$$Val^\omega \left( \sum_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}} = \sum_{(i_j)_{j \in \mathbb{N}} \in I_0 \times I_1 \times \dots} Val^\omega (k_{i_j})_{j \in \mathbb{N}}.$$

Observe that every product  $\omega$ -valuation monoid is a generalized product  $\omega$ -valuation monoid. However, not every generalized product  $\omega$ -valuation monoid is a product  $\omega$ -valuation monoid (see Example 2). We will call the product  $\omega$ -valuation monoid (resp. the generalized product  $\omega$ -valuation monoid)  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  idempotent if the complete monoid  $(K, +, \mathbf{0})$  is idempotent.

Subsequently, we derive properties of product  $\omega$ -valuation monoids and generalized product  $\omega$ -valuation monoids. For simplicity we provide the proofs only for product  $\omega$ -valuation monoids. The reader may check that essentially the same arguments also hold, if  $K$  is a generalized product  $\omega$ -valuation monoid.

**Lemma 1** *Let  $K$  be an idempotent product  $\omega$ -valuation monoid or an idempotent generalized product  $\omega$ -valuation monoid. Then,*

- (i) *[[11], Chapter 5, Lemma 7.3]  $\sum_I \mathbf{1} = \mathbf{1}$  for every set  $I$  with size at most continuum.*
- (ii)  *$\sum_I k = k$  for every set  $I$  with size at most continuum and every  $k \in K$ .*
- (iii)  *$\sum_{k \in K} k + \sum_{k' \in K'} k' = \sum_{k \in K} k$  for every  $K' \subseteq K$ .*
- (iv)  *$\sum_{k'' \in K''} k'' + \sum_{k' \in K'} k' = \sum_{k \in K'+K''} k$  for every  $K', K''$  non-empty subsets of  $K$  with size at most continuum.*

**Proof:**

- (ii) It holds  $\sum_I (k \cdot \mathbf{1}) = k \cdot \sum_I \mathbf{1}$ , for every  $k \in K$ , and index set  $I$ . Hence, by the above property and (i) we get  $\sum_I k = k \cdot \sum_I \mathbf{1} = k \cdot \mathbf{1} = k$ .



(iii) For  $K' = \emptyset$  it is obvious. Otherwise, we get

$$\begin{aligned} \sum_{k \in K} k + \sum_{k' \in K'} k' &= \sum_{k \in K \setminus K'} k + \sum_{k' \in K'} k' + \sum_{k' \in K'} k' \\ &= \sum_{k \in K \setminus K'} k + \sum_{k' \in K'} k' = \sum_{k \in K} k \end{aligned} ,$$

where the first and last equality hold by the completeness axioms of the monoid, and the second one by idempotency.

(iv) For each  $k' \in K'$  (resp.  $k'' \in K''$ ) there exists an index set  $I_{k'} \neq \emptyset$  (resp.  $I_{k''} \neq \emptyset$ ) with size at most continuum such that

$$\begin{aligned} \sum_{k \in K' + K''} k &= \sum_{k \in K' + K''} \left( \sum_{\substack{(k', k'') \in K' \times K'' \\ \text{s.t. } k = k' + k''}} (k' + k'') \right) \\ &= \sum_{k' \in K'} \left( \sum_{I_{k'}} k' \right) + \sum_{k'' \in K''} \left( \sum_{I_{k''}} k'' \right) \\ &= \sum_{k' \in K'} k' + \sum_{k'' \in K''} k'' \end{aligned}$$

where the first and last equality holds by (ii), and the second equality by the completeness axioms of the monoid. □

**Lemma 2** *Let  $K$  be an idempotent product  $\omega$ -valuation monoid or an idempotent generalized product  $\omega$ -valuation monoid, and  $K', K'' \subseteq K$  such that the size of  $K'$  is at most continuum and for every  $k' \in K'$  there exists  $k'' \in K''$  with  $k' \leq k''$ . Then,  $\sum_{k' \in K'} k' \leq \sum_{k'' \in K''} k''$ .*

**Proof:** There exist index sets  $I, J$ , with the size of  $I$  being at most continuum, such that  $K' = \{k'_i \in K \mid i \in I\}$ ,  $K'' = \{k''_j \in K \mid j \in J\}$ . We let

$$\bar{J} = \left\{ j \in J \mid \exists i_j \in I, k'_{i_j} \in K' \text{ such that } k'_{i_j} \leq k''_j \right\}.$$

For every  $j \in \bar{J}$  we let  $I_j = \left\{ i \in I \mid k'_i \leq k''_j \right\}$ . It holds  $\bigcup_{j \in \bar{J}} I_j = I$ . We fix a  $j \in \bar{J}$ . For every  $i \in I_j$ , we have  $k''_j = k'_i + k''_j$  and by idempotency and

Lemma 1(ii) it holds  $k_j'' = \sum_{i \in I_j} (k_i' + k_j'')$ . We thus get

$$\begin{aligned} \sum_{j \in \bar{J}} k_j'' &= \sum_{j \in \bar{J}} \left( \sum_{i \in I_j} (k_i' + k_j'') \right) \\ &= \sum_{j \in \bar{J}} \left( \sum_{i \in I_j} k_i' \right) + \sum_{j \in \bar{J}} \left( \sum_{i \in I_j} k_j'' \right) \\ &= \sum_{i \in I} k_i' + \sum_{j \in \bar{J}} k_j''. \end{aligned}$$

We conclude the second equality by the completeness axioms of the monoid, and the last one by Lemma 1(ii), and the fact that  $I_j$  has size at most continuum for every  $j \in \bar{J}$ . Hence,  $\sum_{i \in I} k_i' \leq \sum_{j \in \bar{J}} k_j'' \leq \sum_{j \in J} k_j''$  where the first

inequality is concluded taking into account the definition of the natural order of  $K$ , and the second inequality holds by Lemma 1(iii), and again by the definition of the natural order of  $K$ .  $\square$

### Lemma 3

(i) Let  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent product  $\omega$ -valuation monoid, and  $L \subseteq_{fin} K$ . If  $(k_i^1)_{i \geq 0}$  and  $(k_i^2)_{i \geq 0}$  are families of elements of  $L$  such that for every  $i \geq 0$ ,  $k_i^1 \leq k_i^2$ , then  $Val^\omega (k_i^1)_{i \geq 0} \leq Val^\omega (k_i^2)_{i \geq 0}$ .

(ii) Let  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent generalized product  $\omega$ -valuation monoid, and  $L \subseteq_{fin} K$ . If  $(k_i^1)_{i \geq 0}$  and  $(k_i^2)_{i \geq 0}$  are families of elements of  $L$  such that for every  $i \geq 0$ ,  $k_i^1 \leq k_i^2$ , and for all but a finite number of  $i \geq 0$ , it holds  $\{k_i^1, k_i^2\} \subseteq L \setminus \{\mathbf{0}, \mathbf{1}\}$ , or  $\{k_i^1, k_i^2\} \subseteq \{\mathbf{0}, \mathbf{1}\}$ , then  $Val^\omega (k_i^1)_{i \geq 0} \leq Val^\omega (k_i^2)_{i \geq 0}$ .

**Proof:**

(i) It holds

$$\begin{aligned} Val^\omega (k_i^2)_{i \geq 0} &= Val^\omega (k_i^2 + k_i^1)_{i \geq 0} = \sum_{(j_0, j_1, \dots) \in \{1, 2\}^\omega} Val^\omega (k_i^{j_i})_{i \geq 0} \\ &= Val^\omega (k_i^1)_{i \geq 0} + \sum_{\substack{(j_0, j_1, \dots) \in \{1, 2\}^\omega \\ (j_0, j_1, \dots) \neq 1^\omega}} Val^\omega (k_i^{j_i})_{i \geq 0} \end{aligned}$$

where the second equality holds by the distributivity of  $Val^\omega$  over finite sums, and the third one by the completeness axioms of the monoid, and this proves our claim.

- (ii) We can prove the claim with the same arguments used in the previous case. □

In the rest of this paper we will consider idempotent product  $\omega$ -valuation monoids  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  (resp. idempotent generalized product  $\omega$ -valuation monoids) that further satisfy the following properties (resp. further satisfy the following properties and the natural order is a total order). For all  $k, k_i \in K, (i \geq 1)$

$$Val^\omega(\mathbf{1}, k_1, k_2, k_3, \dots) = Val^\omega(k_i)_{i \geq 1}, \quad (2)$$

$$k = Val^\omega(k, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots) \quad (3)$$

In the rest of the paper we shall call the properties described by equations 2, 3, **Property 2**, and **Property 3** respectively. We note that Properties 2 and 3 express a notion of neutrality of  $\mathbf{1}$  over  $Val^\omega$ . Next, we present examples of product  $\omega$ -valuation monoids, and generalized product  $\omega$ -valuation monoids.

**Example 1** *Every idempotent totally commutative complete semiring (cf. [23])  $(K, +, \cdot, \mathbf{0}, \mathbf{1})$  can be considered as an idempotent product  $\omega$ -valuation monoid  $(K, +, \cdot, \prod, \mathbf{0}, \mathbf{1})$  if we consider as the  $\omega$ -valuation function the countably infinite products operation  $\prod$  that every totally commutative complete semiring is equipped with. Moreover, these structures satisfy Properties 2, and 3. We can verify these properties, as well as the ones in the definition of product  $\omega$ -valuation monoids in a straightforward way by applying the completeness axioms of the structures.*

**Example 2** *We consider the structure  $(\overline{\mathbb{R}}, \sup, \inf, \liminf, -\infty, \infty)$  where  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$  and  $\liminf$  is an  $\omega$ -valuation function from  $(\overline{\mathbb{R}}_{fin})^\omega$  to  $\overline{\mathbb{R}}$  defined by*

$$\liminf((d_i)_{i \geq 0}) = \begin{cases} -\infty & \text{if there exists } i \geq 0 \text{ with } d_i = -\infty \\ \infty & \text{if for all } i \geq 0, d_i = \infty \\ \lim_{i \geq 0}(\inf \{d_k \mid k \geq i, d_k \neq \infty\}) & \text{if } d_j \neq -\infty \text{ for all } j \geq 0, \\ & \text{and there exist infinitely many } i \geq 0 \text{ with } d_i \neq \infty \\ \inf \{d_i \mid i \geq 0 \text{ with } d_i \neq \infty\} & \text{otherwise} \end{cases}$$

$(\overline{\mathbb{R}}, \sup, \inf, \liminf, -\infty, \infty)$  is an idempotent generalized product  $\omega$ -valuation monoid that satisfy Properties 2, and 3, and the natural order obtained in the structure is a total order. We observe that  $(\overline{\mathbb{R}}, \sup, \inf, \liminf, -\infty, \infty)$  is not a product  $\omega$ -valuation monoid. To verify this observation we present the following counterexample. We consider the families of elements of  $\overline{\mathbb{R}}$ ,  $(k_{i_j})_{i_j \in I_j}$ , where for every  $j \geq 0$  with  $j \neq 1$ , it holds  $I_j = \{1, 2\}$ , and  $k_1 = \infty, k_2 = 6$ , and for  $j = 1$ , we have  $I_1 = \{1\}$ , and  $k_1 = 5$ . Then,

$$\liminf_{j \in \mathbb{N}} \left( \sup_{i_j \in I_j} k_{i_j} \right) = 5, \text{ and } \sup_{(i_j)_{j \in \mathbb{N}} \in I_0 \times I_1 \times \dots} \left( \liminf_{j \in \mathbb{N}} (k_{i_j}) \right) = 6.$$

In the Appendix we prove that the structure presented in the previous example is indeed a generalized product  $\omega$ -valuation monoid. In [13] the authors have also considered an  $\omega$ -product valuation monoid where the classical  $\liminf$ -function is used. The definition of  $\liminf$  in our example is motivated by the need to capture the semantics of weighted logics that will be introduced of Section 6. In particular, the semantics of our *until*-operator expresses the fact that whenever the  $\omega$ -valuation function is applied, then the valuation should take into account only a finite number of first terms of an infinite sequence. This in our example is expressed by the fourth case in the definition of the  $\omega$ -valuation function.

## 5 Weighted Generalized Büchi Automata with $\varepsilon$ -transitions over Product $\omega$ -valuation Monoids, and Generalized Product $\omega$ -valuation Monoids

Let  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent product  $\omega$ -valuation monoid, and  $A$  be an alphabet. We introduce now the models of weighted generalized Büchi automata with  $\varepsilon$ -transitions and weighted Büchi automata with  $\varepsilon$ -transitions over  $A$ , and  $K$ . We note that weighted Büchi automata over  $\omega$ -valuation monoids have already been considered in [13, 24].

### Definition 3

- (i) A weighted generalized Büchi automaton with  $\varepsilon$ -transitions ( $\varepsilon$ -wgBa for short) over  $A$  and  $K$  is a quadruple  $\mathcal{A} = (Q, wt, I, \mathcal{F})$ , where  $Q$  is the finite set of states,  $wt : Q \times (A \cup \{\varepsilon\}) \times Q \rightarrow K$  is a mapping assigning weights to the transitions of the automaton,  $I$  is the set of initial states and  $\mathcal{F} = \{F_1, \dots, F_l\}$  is the set of final sets  $F_i \in \mathcal{P}(Q)$ ,

for every  $1 \leq i \leq l$ . For every  $t \in Q \times \{\varepsilon\} \times Q$  we require that  $wt(t) = \mathbf{0}$  or  $wt(t) = \mathbf{1}$ . Moreover, for every  $(q, \varepsilon, q') \in Q \times \{\varepsilon\} \times Q$  with  $wt(q, \varepsilon, q') = \mathbf{1}$ , and every  $i \in \{1, \dots, l\}$ , we have  $q \in F_i$  iff  $q' \in F_i$ .

- (ii) An  $\varepsilon$ -wgBa is a weighted Büchi automaton with  $\varepsilon$ -transitions ( $\varepsilon$ -wBa for short) if  $l = 1$ , i.e., there is only one final set.
- (iii) An  $\varepsilon$ -wBa is a weighted Büchi automaton (wBa for short) if for every  $q, q' \in Q$  it holds  $wt(q, \varepsilon, q') = \mathbf{0}$ .

If  $\mathcal{A} = (Q, wt, I, \mathcal{F})$  is an  $\varepsilon$ -wBa, then we simply write  $\mathcal{A} = (Q, wt, I, F)$ . Let  $w = a_0 a_1 \dots \in A^\omega$  with  $a_i \in A$  ( $i \geq 0$ ). A path  $P_w$  of  $\mathcal{A}$  over  $w$  is an infinite sequence of transitions  $P_w = (q_j, b_j, q_{j+1})_{j \geq 0}$ ,  $b_j \in A \cup \{\varepsilon\}$  ( $j \geq 0$ ), such that  $w = b_0 b_1 \dots$ . Let  $i_0 \leq i_1 \leq i_2 \leq i_3 \leq \dots$  be the sequence of positions with  $b_{i_k} = a_k$  for every  $k \geq 0$ , and  $h_0 \leq h_1 \leq h_2 \leq h_3 \leq \dots$  be the sequence of positions with  $b_{h_l} = \varepsilon$  for every  $l \geq 0$ . Then, we let the weight of  $P_w$  be the value

$$weight_{\mathcal{A}}(P_w) = \begin{cases} Val^\omega (wt(q_{i_k}, a_k, q_{i_k+1}))_{k \geq 0} & \text{if } wt(q_{h_l}, \varepsilon, q_{h_l+1}) = \mathbf{1} \\ & \text{for every } l \geq 0 \\ \mathbf{0} & \text{otherwise} \end{cases}$$

Let  $P_w = (q_j, b_j, q_{j+1})_{j \geq 0}$ ,  $b_j \in A \cup \{\varepsilon\}$  ( $j \geq 0$ ) be a path of  $\mathcal{A}$  over  $w$ . The set of states that appear infinitely often along  $P_w$  is denoted by  $In^Q(P_w)$ . The path  $P_w$  is called *successful* if  $q_0 \in I$  and  $In^Q(P_w) \cap F_i \neq \emptyset$ , for every  $i \in \{1, \dots, l\}$ . We shall denote by  $succ_{\mathcal{A}}(w)$  the set of all successful paths of  $\mathcal{A}$  over  $w$ . The behavior of  $\mathcal{A}$  is the infinitary series  $\|\mathcal{A}\| : A^\omega \rightarrow K$  with coefficients specified, for every  $w \in A^\omega$ ,

$$(\|\mathcal{A}\|, w) = \sum_{P_w \in succ_{\mathcal{A}}(w)} weight_{\mathcal{A}}(P_w).$$

**Remark 2** In the definition of our  $\varepsilon$ -wgBa we impose a restriction on the weights that  $wt$  assigns to  $\varepsilon$ -transitions. More specifically, we require that  $\varepsilon$ -transitions have weight  $\mathbf{0}$ , or  $\mathbf{1}$ , and that  $\mathbf{1}$ -weight  $\varepsilon$ -transitions are only allowed between states that belong to the same final subsets of the automaton. As it will be presented in the sequel, this restriction is sufficient for expressing the intuition of the translation of our weighted formulas to  $\varepsilon$ -wgBa, as we will need transitions that reduce formulas without modifying the weight of the

path, and in the same time respecting the conditions imposed by final subsets. In the general framework, with this definition we obtain a generalization of  $wBa$  that allows flexibility to move between states, and at the same time respects acceptance conditions, and weight computation that are determined by transitions that consume a letter of the input word.

**Remark 3** As mentioned before,  $wBa$  over  $\omega$ -valuation monoids have already been considered in [13, 24]. In contrast to [13, 24], in our notations we do not explicitly define a set of transitions as a subset of all possible triples  $(q, a, q')$ , and then use a weight function to assign weights to the elements of this subset. Our weight function  $w$  assigns weights to all possible transitions of the automaton, and then, similarly to [13, 24], we obtain the weight of a path by applying  $Val^\omega$  to the sequence of weights of the transitions of the path. By the definition of  $Val^\omega$ , if the weight of one transition is  $\mathbf{0}$ , then the weight of a path is  $\mathbf{0}$ . This implies that given a  $wBa$  defined by Definition 3, we can obtain an equivalent  $wBa$  defined as in [13] with a set of transitions at least the ones with non-zero weight at the original automaton, and vice-versa. Given a  $wBa$  defined as in [13], we can construct a  $wBa$  defined by Definition 3, by assigning the weight  $\mathbf{0}$  to every tuple  $(q, a, q')$  that does not belong to the set of transitions of the original  $wBa$ . Hence the two notations lead to equivalent definitions. We note that for [24] we refer to the simplest form of  $wBa$  introduced in that paper.

Two  $\varepsilon$ - $wgBa$  are called equivalent if they have the same behavior. We shall also denote an  $\varepsilon$ -transition with weight  $\mathbf{1}$  by  $\xrightarrow{\varepsilon}$  and we will write  $\xrightarrow{*}$  for the transitive and reflexive closure of  $\xrightarrow{\varepsilon}$ . Finally, for every  $w = a_0 \dots a_n \in A^+$  we shall denote by  $q \xrightarrow{w} q'$  a sequence of transitions  $(q_j, a_j, q_{j+1})_{0 \leq j \leq n}$  with  $q_0 = q$ , and  $q_{n+1} = q'$ . Now, we let

$$pri_{\mathcal{A}}(w) = \{weight_{\mathcal{A}}(P_w) \mid P_w \in succ_{\mathcal{A}}(w)\}$$

for every  $w \in A^\omega$ .

$\varepsilon$ - $wgBa$  (resp.  $\varepsilon$ - $wBa$ , and  $wBa$ ) over generalized product  $\omega$ -valuation monoids are defined in the same way with  $\varepsilon$ - $wgBa$  (resp.  $\varepsilon$ - $wBa$ , and  $wBa$ ) over product  $\omega$ -valuation monoids presented above.

**Lemma 4** Let  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent product  $\omega$ -valuation monoid or an idempotent generalized product  $\omega$ -valuation monoid. For every  $\varepsilon$ - $wgBa$  over  $A$  and  $K$  we can effectively construct an equivalent  $\varepsilon$ - $wBa$ .

**Proof:** Let  $\mathcal{A} = (Q, wt, I, \mathcal{F})$  be an  $\varepsilon$ -wgBa over  $A$  and  $K$  with  $\mathcal{F} = \{F_1, \dots, F_l\}$ . We let  $\mathcal{A}' = (Q', I', wt', F')$  be an  $\varepsilon$ -wBa defined as follows:

- $Q' = Q \times \{1, \dots, l\}$ ,
- $I' = I \times \{1\}$ ,
- For every  $((q, i), b, (q', j)) \in Q' \times (A \cup \{\varepsilon\}) \times Q'$  we let:
$$wt'((q, i), b, (q', j)) = \begin{cases} & \text{if } (b \in A, i = j \text{ and } q \notin F_i) \\ wt(q, b, q') & \text{or } (b \in A, j \equiv (i + 1) \bmod l \text{ and } q \in F_i) \\ \mathbf{0} & \text{or } (b = \varepsilon \text{ and } i = j) \\ & \text{otherwise} \end{cases},$$
- $F' = F_1 \times \{1\}$ .

We will prove that  $pri_{\mathcal{A}'}(w) \setminus \{\mathbf{0}\} = pri_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$  for every  $w \in A^\omega$ . To this end let  $w = a_0a_1 \dots \in A^\omega$  and  $P_w = (q_i, b_i, q_{i+1})_{i \geq 0}$  be a successful path of  $\mathcal{A}$  over  $w$  with  $weight_{\mathcal{A}}(P_w) \neq \mathbf{0}$ . Moreover, let  $i_0 \leq i_1 \leq i_2 \leq i_3 \leq \dots$  be the sequence of positions with  $b_{i_k} = a_k$  for every  $k \geq 0$ , and  $h_0 \leq h_1 \leq h_2 \leq h_3 \leq \dots$  be the sequence of positions with  $b_{h_l} = \varepsilon$  for every  $l \geq 0$ . It holds

$$Val^\omega(wt(q_{i_k}, a_k, q_{i_k+1}))_{k \geq 0} \neq \mathbf{0} \text{ and } wt(q_{h_k}, \varepsilon, q_{h_k+1}) = \mathbf{1} \text{ for all } k \geq 0.$$

We define the path

$$P'_w = ((q_i, g_i), b_i, (q_{i+1}, g_{i+1}))_{i \geq 0}$$

of  $\mathcal{A}'$  over  $w$  by setting  $g_0 = 1$  and for every  $i \geq 1$ , we point out the following cases. If  $b_{i-1} \neq \varepsilon$ , we let  $g_i = g_{i-1}$  if  $q_{i-1} \notin F_{g_{i-1}}$ , and  $g_i \equiv (g_{i-1} + 1) \bmod l$  if  $q_{i-1} \in F_{g_{i-1}}$ . If  $b_{i-1} = \varepsilon$ , we let  $g_i = g_{i-1}$ . By construction  $P'_w$  is successful and for every  $i \geq 0$  it holds

$$wt'((q_i, g_i), b_i, (q_{i+1}, g_{i+1})) = wt(q_i, b_i, q_{i+1})$$

which implies that

$$\begin{aligned} & weight_{\mathcal{A}'}(P'_w) \\ &= Val^\omega(wt'((q_{i_k}, g_{i_k}), a_k, (q_{i_k+1}, g_{i_k+1})))_{k \geq 0} \\ &= Val^\omega(wt(q_{i_k}, a_k, q_{i_k+1}))_{k \geq 0} \\ &= weight_{\mathcal{A}}(P_w). \end{aligned}$$

We thus get  $\text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\} \subseteq \text{pri}_{\mathcal{A}'}(w) \setminus \{\mathbf{0}\}$ . In order to prove the opposite inclusion we let  $P'_w = ((q_i, g_i), b_i, (q_{i+1}, g_{i+1}))_{i \geq 0}$  be a successful path of  $\mathcal{A}'$  over  $w$  with  $\text{weight}_{\mathcal{A}'}(P'_w) \neq \mathbf{0}$ , which by construction of  $\mathcal{A}'$  implies  $\text{wt}'((q_i, g_i), b_i, (q_{i+1}, g_{i+1})) = \text{wt}(q_i, b_i, q_{i+1})$  and either  $b_i \in A, g_i = g_{i+1}$  and  $q_i \notin F_{g_i}$ , or  $b_i \in A, g_{i+1} = (g_i + 1) \bmod l$  and  $q_i \in F_{g_i}$ , or  $b_i = \varepsilon$  and  $g_i = g_{i+1}$ . Then,  $P_w = (q_i, b_i, q_{i+1})_{i \geq 0}$  is a successful path of  $\mathcal{A}$  over  $w$  and

$$\begin{aligned} \text{weight}_{\mathcal{A}'}(P'_w) &= \text{Val}^\omega(\text{wt}'((q_{i_k}, g_{i_k}), a_k, (q_{i_k+1}, g_{i_k+1})))_{k \geq 0} \\ &= \text{Val}^\omega(\text{wt}(q_{i_k}, a_k, q_{i_k+1}))_{k \geq 0} \\ &= \text{weight}_{\mathcal{A}}(P_w) \end{aligned}$$

where the sequence of positions  $i_0 \leq i_1 \leq i_2 \leq i_3 \leq \dots$  is defined as before. Thus,  $\text{pri}_{\mathcal{A}'}(w) \setminus \{\mathbf{0}\} \subseteq \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$ . Which implies that

$$\text{pri}_{\mathcal{A}'}(w) \setminus \{\mathbf{0}\} = \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}.$$

Hence,

$$\begin{aligned} (\|\mathcal{A}'\|, w) &= \sum_{k \in \text{pri}_{\mathcal{A}'}(w)} k = \sum_{k \in \text{pri}_{\mathcal{A}'}(w) \setminus \{\mathbf{0}\}} k \\ &= \sum_{k \in \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}} k = \sum_{k \in \text{pri}_{\mathcal{A}}(w)} k = (\|\mathcal{A}\|, w), \end{aligned}$$

where the first and last equality hold due to the completeness axioms of the monoid and Lemma 1(ii), and this concludes our proof. For idempotent generalized product  $\omega$ -valuation monoids we can prove the lemma's claim using the same arguments.  $\square$

We shall need some auxiliary definitions. Let  $\mathcal{A} = (Q, \text{wt}, I, F)$  be an  $\varepsilon$ -wBa over  $A$  and  $K$ ,  $w = a_0 a_1 \dots \in A^\omega$  and  $P_w = (q_i, a_i, q_{i+1})_{i \geq 0} \in \text{succ}_{\mathcal{A}}(w)$  with no  $\varepsilon$ -transitions. We consider the set of paths  $\text{Paths}(P_w) \subseteq \text{succ}_{\mathcal{A}}(w)$  containing  $P_w$  and every path derived by  $P_w$  if we replace one or more transitions  $(q_i, a_i, q_{i+1}), i \geq 0$ , by a sequence of transitions of the form  $q_i \xrightarrow{*} \tilde{q} \xrightarrow{a_i} \bar{q} \xrightarrow{*} q_{i+1}$ . Furthermore, we let

$$\mathcal{V}_{P_w} = \left\{ k \in K \mid \exists \tilde{P}_w \in \text{Paths}(P_w) \text{ with } \text{weight}_{\mathcal{A}}(\tilde{P}_w) = k \right\}.$$

### Lemma 5

- (i) Let  $(K, +, \cdot, \text{Val}^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent product  $\omega$ -valuation monoid. For every  $\varepsilon$ -wBa over  $A$  and  $K$  we can effectively construct an equivalent wBa over  $A$  and  $K$ .



(ii) Let  $(K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent generalized product  $\omega$ -valuation monoid. For every  $\varepsilon$ -wBa over  $A$  and  $K$  we can effectively construct an equivalent wBa over  $A$  and  $K$ .

**Proof:**

(i) Let  $\mathcal{A} = (Q, wt, I, F)$  be an  $\varepsilon$ -wBa over  $A$  and  $K$ . We define the wBa  $\mathcal{A}' = (Q, wt', I, F)$  by setting

$$wt'(q, a, q') = \sum_{\substack{\tilde{q}, \bar{q} \in Q \\ q \xrightarrow{*} \tilde{q}, \bar{q} \xrightarrow{*} q'}} wt(\tilde{q}, a, \bar{q}).$$

We let  $w = a_0 a_1 \dots \in A^\omega$  and  $P'_w = (q_i, a_i, q_{i+1})_{i \geq 0} \in succ_{\mathcal{A}'}(w)$ . For all  $i \geq 0$  it holds  $wt'(q_i, a_i, q_{i+1}) = \sum_{\substack{\tilde{q}_i, \bar{q}_i \in Q \\ q_i \xrightarrow{*} \tilde{q}_i, \bar{q}_i \xrightarrow{*} q_{i+1}}} wt(\tilde{q}_i, a_i, \bar{q}_i)$ . Then,

$$\begin{aligned} weight_{\mathcal{A}'}(P'_w) &= Val^\omega \left( \sum_{\substack{\tilde{q}_i, \bar{q}_i \in Q \\ q_i \xrightarrow{*} \tilde{q}_i, \bar{q}_i \xrightarrow{*} q_{i+1}}} wt(\tilde{q}_i, a_i, \bar{q}_i) \right)_{i \geq 0} \\ &= \sum_{\substack{i \geq 0 \\ \tilde{q}_i, \bar{q}_i \in Q \\ q_i \xrightarrow{*} \tilde{q}_i, \bar{q}_i \xrightarrow{*} q_{i+1}}} Val^\omega (wt(\tilde{q}_i, a_i, \bar{q}_i))_{i \geq 0} \end{aligned}$$

where the second equality holds by the distributivity of  $Val^\omega$  over finite sums. Clearly,  $P'_w$  is also a successful path of  $\mathcal{A}$  over  $w$  and it holds  $weight_{\mathcal{A}'}(P'_w) = \sum_{\tilde{P}_w \in Paths(P'_w)} weight_{\mathcal{A}}(\tilde{P}_w) = \sum_{k \in \mathcal{V}_{P'_w}} k$ . The last equality is concluded by the completeness axioms of the monoid and Lemma 1(ii). Hence, we get

$$(\|\mathcal{A}'\|, w) = \sum_{P'_w \in succ_{\mathcal{A}'}(w)} \left( \sum_{k \in \mathcal{V}_{P'_w}} k \right) = \sum_{P'_w \in succ_{\mathcal{A}'}(w)} \left( \sum_{k \in \mathcal{V}_{P'_w} \setminus \{\mathbf{0}\}} k \right).$$

We show that  $\left(\bigcup_{P'_w \in \text{succ}_{\mathcal{A}'}(w)} \mathcal{V}_{P'_w}\right) \setminus \{\mathbf{0}\} = \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$ . The first inclusion  $\left(\bigcup_{P'_w \in \text{succ}_{\mathcal{A}'}(w)} \mathcal{V}_{P'_w}\right) \setminus \{\mathbf{0}\} \subseteq \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$  holds by definition of  $\mathcal{V}_{P'_w}$ . To prove the converse inclusion, i.e.,  $\text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\} \subseteq \left(\bigcup_{P'_w \in \text{succ}_{\mathcal{A}'}(w)} \mathcal{V}_{P'_w}\right) \setminus \{\mathbf{0}\}$ , we prove that for every  $k \in \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$  there exists  $P'_w \in \text{succ}_{\mathcal{A}'}(w)$  such that  $k \in \mathcal{V}_{P'_w}$ . To this end we fix a  $k \in \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$  and we let

$$P_w : q_0 \xrightarrow{*} q_{i_1} \xrightarrow{w(0)\dots w(k_1-1)} q_{i_1+k_1} \xrightarrow{*} q_{i_2} \xrightarrow{w(k_1)\dots w(k_1+k_2-1)} q_{i_2+k_2} \xrightarrow{*} q_{i_3} \xrightarrow{w(k_1+k_2)\dots w(k_1+k_2+k_3-1)} q_{i_3+k_3} \xrightarrow{*} \dots$$

be a successful path of  $\mathcal{A}$  over  $w$  with  $\text{weight}_{\mathcal{A}}(P_w) = k, k_j \in \mathbb{N}, j \geq 1$ . We define the path  $P'_w$  of  $\mathcal{A}'$  by setting

$$P'_w : q_0 \xrightarrow{w(0)} q_{i_1+1} \xrightarrow{w(1)} q_{i_1+2} \dots \\ \dots q_{i_1+k_1-1} \xrightarrow{w(k_1-1)} q_{i_1+k_1} \xrightarrow{w(k_1)} q_{i_2+1} \dots \\ \dots q_{i_2+k_2-1} \xrightarrow{w(k_1+k_2-1)} q_{i_2+k_2} \dots$$

By Definition 3 we get that  $P'_w \in \text{succ}_{\mathcal{A}'}(w)$  (observe that for every  $j \geq 0, q_{i_j+k_j} \in F \Leftrightarrow q_{i_{j+1}} \in F$ , where  $k_0 = i_0 = 0$ ). Moreover,  $P'_w \in \text{succ}_{\mathcal{A}}(w)$  and  $\text{weight}_{\mathcal{A}}(P_w) \in \mathcal{V}_{P'_w}$  as wanted. Hence,

$$\begin{aligned} (\|\mathcal{A}'\|, w) &= \sum_{P'_w \in \text{succ}(\mathcal{A}')} \left( \sum_{k \in \mathcal{V}_{P'_w} \setminus \{\mathbf{0}\}} k \right) \\ &= \sum_{k \in \text{pri}_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}} k = \sum_{k \in \text{pri}_{\mathcal{A}}(w)} k = (\|\mathcal{A}\|, w) \end{aligned}$$

and the proof is completed.

- (ii) Let  $\mathcal{A} = (Q, wt, I, F)$  be an  $\varepsilon$ -wBa over  $A$  and  $K$ . We define the wBa  $\overline{\mathcal{A}} = (\overline{Q}, \overline{wt}, \overline{I}, \overline{F})$  by setting  $\overline{Q} = Q \cup S$  where  $S = \{s_q \mid q \in Q\}$

is a set of new states. Moreover, we let  $\bar{I} = \{s_q \mid q \in I\} \cup I$ ,  $\bar{F} = \{s_q \mid q \in F\} \cup F$ , and for every  $(p, a, \bar{p}) \in \bar{Q} \times A \times \bar{Q}$  we set

$$\overline{wt}(p, a, \bar{p}) = \begin{cases} \mathbf{1} & \begin{array}{l} \text{if } p = s_q, \bar{p} = s_{\bar{q}} \in S \\ \text{and there exist } \tilde{q}, q' \in Q, \\ \text{with } wt(\tilde{q}, a, q') = \mathbf{1}, \\ q \xrightarrow{*} \tilde{q}, q' \xrightarrow{*} \bar{q} \end{array} \\ \sum_{\substack{\tilde{q}, \bar{q} \in Q \\ q_1 \xrightarrow{*} \tilde{q}, \bar{q} \xrightarrow{*} q_2 \\ wt(\tilde{q}, a, \bar{q}) \neq \mathbf{1}, \mathbf{0}}} wt(\tilde{q}, a, \bar{q}) & \begin{array}{l} \text{if } q_1 = p, q_2 = \bar{p} \in Q, \text{ or if} \\ p = q_1 \in Q, \bar{p} = s_{q_2} \in S, \text{ or if} \\ p = s_{q_1} \in S, \bar{p} = q_2 \in Q \end{array} \\ \mathbf{0} & \text{otherwise} \end{cases} .$$

We let  $w = a_0 a_1 \dots \in A^\omega$  and  $P'_w = (q_i, a_i, q_{i+1})_{i \geq 0} \in succ_{\bar{\mathcal{A}}}(w)$ , with non-zero weight, such that  $q_i \in Q$  for all  $i \geq 0$ . Then,

$$\begin{aligned} weight_{\bar{\mathcal{A}}}(P'_w) &= Val^\omega \left( \sum_{\substack{\tilde{q}_i, \bar{q}_i \in Q \\ q_i \xrightarrow{*} \tilde{q}_i, \bar{q}_i \xrightarrow{*} q_{i+1} \\ wt(\tilde{q}_i, a, \bar{q}_i) \neq \mathbf{1}, \mathbf{0}}} wt(\tilde{q}_i, a_i, \bar{q}_i) \right)_{i \geq 0} \\ &= \sum_{\substack{\tilde{q}_i, \bar{q}_i \in Q \\ q_i \xrightarrow{*} \tilde{q}_i, \bar{q}_i \xrightarrow{*} q_{i+1} \\ wt(\tilde{q}_i, a, \bar{q}_i) \neq \mathbf{1}, \mathbf{0}}} Val^\omega (wt(\tilde{q}_i, a_i, \bar{q}_i))_{i \geq 0} \end{aligned}$$

where the second equality is obtained by the distributivity of  $Val^\omega$  over finite sums for generalized product  $\omega$ -valuation monoids. For every  $P'_w = (q_i, a_i, q_{i+1})_{i \geq 0} \in succ_{\bar{\mathcal{A}}}(w)$  such that  $q_i \in Q$  for all  $i \geq 0$ , we consider the set of paths  $\overline{Paths}(P'_w)$  that contains all the paths derived by  $P'_w$  if we replace one or more states in  $P'_w$  by its decoy in  $S$ . Clearly, every path in  $\overline{Paths}(P'_w)$  is a successful path of  $\bar{\mathcal{A}}$  over  $w$  and it holds

$$weight_{\bar{\mathcal{A}}}(P'_w) + \sum_{\bar{P}_w \in \overline{Paths}(P'_w)} weight_{\bar{\mathcal{A}}}(\bar{P}_w) = \sum_{k \in \mathcal{V}_{P'_w}} k.$$

Hence, we get

$$\begin{aligned}
& (\|\overline{\mathcal{A}}\|, w) \\
&= \sum_{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega} weight_{\overline{\mathcal{A}}}(P'_w) \\
&+ \sum_{\substack{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega \\ \overline{P}_w \in \overline{Paths}(P'_w)}} weight_{\overline{\mathcal{A}}}(\overline{P}_w) \\
&= \sum_{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega} \left( weight_{\overline{\mathcal{A}}}(P'_w) + \sum_{\overline{P}_w \in \overline{Paths}(P'_w)} weight_{\overline{\mathcal{A}}}(\overline{P}_w) \right) \\
&= \sum_{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega} \left( \sum_{k \in \mathcal{V}_{P'_w}} k \right) \\
&= \sum_{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega} \left( \sum_{k \in \mathcal{V}_{P'_w} \setminus \{\mathbf{0}\}} k \right).
\end{aligned}$$

We note that the second equality holds by the completeness axioms of the monoid. Using the same arguments as in case (i), and taking into account that for every  $k \in pri_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$  there exists  $P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega$  such that  $k \in \mathcal{V}_{P'_w}$ , we can prove that  $\left( \bigcup_{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega} \mathcal{V}_{P'_w} \right) \setminus \{\mathbf{0}\} = pri_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}$ , and thus it holds

$$\begin{aligned}
(\|\overline{\mathcal{A}}\|, w) &= \sum_{P'_w \in succ_{\overline{\mathcal{A}}}(w) \cap (Q \times A \times Q)^\omega} \left( \sum_{k \in \mathcal{V}_{P'_w} \setminus \{\mathbf{0}\}} k \right) \\
&= \sum_{k \in pri_{\mathcal{A}}(w) \setminus \{\mathbf{0}\}} k = \sum_{k \in pri_{\mathcal{A}}(w)} k = (\|\mathcal{A}\|, w)
\end{aligned}$$

as wanted. □

## 6 Weighted LTL over Product $\omega$ -valuation Monoids

In what follows, we present our definition of the weighted *LTL* over product  $\omega$ -valuation monoids. We recall that a weighted *LTL* has appeared for the first time in [19]. This follows the definition of weighted *MSO* logic over semirings presented in [10]. We also recall that a weighted *MSO* logic over  $\omega$ -valuation monoids was defined in [13], where the  $\omega$ -valuation function has been used for the definition of the semantics of the universal first order, and second order quantifier. Analogously, we will use the  $\omega$ -valuation function of the underlying structure for the definition of the semantics of the *always* operator of our logic.

Let  $AP$  be a finite set of atomic propositions and  $K = (K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent product  $\omega$ -valuation monoid. In the sequel we shall denote the elements of  $AP$  by  $a, b, c, \dots$ . The syntax of the weighted *LTL* over  $AP$  and  $K$  is given by the grammar

$$\varphi ::= k \mid a \mid \neg a \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \bigcirc \varphi \mid \varphi U \psi \mid \Box \varphi$$

where  $k \in K$  and  $a \in AP$ .

We denote by  $LTL(K, AP)$  the class of all weighted *LTL*-formulas over  $AP$  and  $K$ .

**Definition 4** *The semantics  $\|\varphi\|$  of formulas  $\varphi \in LTL(K, AP)$  are represented as infinitary series in  $K \langle\langle (\mathcal{P}(AP))^\omega \rangle\rangle$  inductively defined in the following way. For every  $w \in (\mathcal{P}(AP))^\omega$  we set*

- $(\|k\|, w) = k,$
- $(\|a\|, w) = \begin{cases} \mathbf{1} & \text{if } a \in w(0) \\ \mathbf{0} & \text{otherwise} \end{cases},$
- $(\|\neg a\|, w) = \begin{cases} \mathbf{1} & \text{if } a \notin w(0) \\ \mathbf{0} & \text{otherwise} \end{cases},$
- $(\|\varphi \wedge \psi\|, w) = (\|\varphi\|, w) \cdot (\|\psi\|, w),$
- $(\|\varphi \vee \psi\|, w) = (\|\varphi\|, w) + (\|\psi\|, w),$
- $(\|\bigcirc \varphi\|, w) = (\|\varphi\|, w_{\geq 1}),$
- $(\|\varphi U \psi\|, w) = \sum_{i \geq 0} Val^\omega((\|\varphi\|, w_{\geq 0}), \dots, (\|\varphi\|, w_{\geq i-1}), (\|\psi\|, w_{\geq i}), \mathbf{1}, \mathbf{1}, \dots),$
- $(\|\Box \varphi\|, w) = Val^\omega((\|\varphi\|, w_{\geq i}))_{i \geq 0}.$

We shall denote by *true* the formula  $\mathbf{1} \in K$ . The syntactic boolean fragment  $bLTL(K, AP)$  of  $LTL(K, AP)$  is given by the grammar

$$\varphi ::= \mathbf{0} \mid true \mid a \mid \neg a \mid \varphi \vee \psi \mid \varphi \wedge \psi \mid \bigcirc \varphi \mid \varphi U \psi \mid \square \varphi$$

where  $a \in AP$ . Inductively, we can prove that  $\text{Im}(\|\varphi\|) \subseteq \{\mathbf{0}, \mathbf{1}\}$  for every  $\varphi \in bLTL(K, AP)$  and the semantics of the formulas of  $bLTL(K, AP)$  and the corresponding classical  $LTL$  formulas coincide. Let  $\varphi, \psi \in LTL(K, AP)$ . We will call  $\varphi, \psi$  equivalent if  $(\|\varphi\|, w) = (\|\psi\|, w)$  for every  $w \in (\mathcal{P}(AP))^\omega$ .

**Proposition 1** *For every  $\varphi, \psi \in LTL(K, AP)$  the following equivalences hold:*

- $\psi \wedge \psi \equiv \psi$ , whenever  $\psi$  is boolean
- $\varphi \wedge \psi \equiv \psi \wedge \varphi$ , whenever  $\varphi$  is boolean
- $\varphi \wedge true \equiv \varphi$
- $\bigcirc(\varphi \wedge \psi) \equiv (\bigcirc\varphi) \wedge (\bigcirc\psi)$
- $\bigcirc(\varphi \vee \psi) \equiv (\bigcirc\varphi) \vee (\bigcirc\psi)$
- $\bigcirc(\varphi U \psi) \equiv (\bigcirc\varphi) U (\bigcirc\psi)$
- $\bigcirc(\square\varphi) \equiv \square(\bigcirc\varphi)$
- $\bigcirc k \equiv k$ , for all  $k \in K$

As in [21] we let an *LTL-step formula* be an  $LTL(K, AP)$ -formula of the form  $\bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i)$  with  $k_i \in K$  and  $\varphi_i \in bLTL(K, AP)$  for every  $1 \leq i \leq n$ .

We denote by  $stLTL(K, AP)$  the class of all *LTL-step formulas* over  $AP$  and  $K$ . We introduce now the syntactic fragment of *restricted U-nesting LTL-formulas*.

**Definition 5** *The fragment of restricted U-nesting LTL-formulas over  $AP$  and  $K$ , denoted by  $RULTL(K, AP)$ , is the least class of formulas in  $LTL(K, AP)$  which is defined inductively in the following way.*

- $k \in RULTL(K, AP)$  for every  $k \in K$ .
- $bLTL(K, AP) \subseteq RULTL(K, AP)$ .

- If  $\varphi \in RULTL(K, AP)$ , then  $\bigcirc\varphi \in RULTL(K, AP)$ .
- If  $\varphi, \psi \in RULTL(K, AP)$ , then  $\varphi \vee \psi \in RULTL(K, AP)$ .
- If  $\varphi \in bLTL(K, AP)$  and  $\psi \in RULTL(K, AP)$ ,  
then  $\varphi \wedge \psi, \psi \wedge \varphi \in RULTL(K, AP)$ .
- If  $\varphi, \psi \in stLTL(K, AP)$ , then  $\varphi U \psi \in RULTL(K, AP)$ .
- If  $\varphi \in stLTL(K, AP)$ , then  $\square\varphi \in RULTL(K, AP)$ .

**Remark 4** Let  $\varphi \in LTL(K, AP)$ . We will say that  $\varphi$  is of form A, if it is of the form  $\varphi = \bigwedge_{1 \leq i \leq n} \varphi_i$ ,  $n \geq 1$ , where there exist at most one  $i \in \{1, \dots, n\}$  such that  $\varphi_i \notin bLTL(K, AP)$ , and for all  $i \in \{1, \dots, n\}$ ,  $\varphi_i$  is not a finite conjunction  $\lambda_1 \wedge \dots \wedge \lambda_k$  with  $k \geq 2$ . We can prove inductively in the structure of  $RULTL(K, AP)$ -formulas that every  $\varphi \in RULTL(K, AP)$  is of form A. For  $\varphi = k$ ,  $\varphi = a$ ,  $\varphi = \neg a$ ,  $\varphi = \bigcirc\varphi'$ ,  $\varphi = \varphi' \vee \varphi''$ ,  $\varphi = \psi U \xi$ ,  $\varphi = \square\psi$  where  $\varphi', \varphi'' \in RULTL(K, AP)$ ,  $\xi, \psi \in stLTL(K, AP)$ , we have  $\varphi = \bigwedge_{1 \leq i \leq 1} \varphi_i$  with  $\varphi_1 = \varphi$ . Assume that  $\varphi = \psi \wedge \xi$  where  $\psi = \bigwedge_{1 \leq i \leq n} \psi_i \in RULTL(K, AP)$ , and  $\xi = \bigwedge_{1 \leq j \leq m} \xi_j \in bLTL(K, AP)$  are in form A. Then,  $\varphi = \bigwedge_{1 \leq i \leq m+n} \varphi_i$ , where  $\varphi_i = \psi_i$ ,  $1 \leq i \leq n$ , and  $\varphi_{n+j} = \xi_j$ ,  $1 \leq j \leq m$ , is also in form A, i.e.,  $\varphi_i$  is not a conjunction  $\lambda_1 \wedge \dots \wedge \lambda_k$  with  $k \geq 2$  for all  $i \in \{1, \dots, m+n\}$ , and since  $\xi$  is boolean, there exists at most an  $i \in \{1, \dots, n+m\}$  with  $\varphi_i \notin bLTL(K, AP)$ .

A formula  $\varphi \in LTL(K, AP)$  is called *reduced* if (a) for every subformula of the form  $\varphi_1 \wedge \dots \wedge \varphi_k$  with  $k \geq 2$  it holds:  $\varphi_i \neq true$  for every  $1 \leq i \leq k$ , and  $\varphi_i \neq \varphi_j$  whenever  $\varphi_i, \varphi_j$  with  $1 \leq i < j \leq k$  are boolean and (b) no until operator is in the scope of any next operator. For every  $\varphi \in LTL(K, AP)$  we can effectively construct an equivalent reduced formula by applying the equivalences of Proposition 1. We shall denote this formula by  $\varphi_{re}$ .

In the sequel, we prove that for every reduced restricted  $U$ -nesting formula  $\varphi$  there exists an  $\varepsilon$ -wgBa accepting its semantics. We recall that the value assigned by  $\|\varphi\|$  to an infinite word  $w$  is computed by induction on the structure of  $\varphi$ . Moreover, in the induction for the semantics of the  $\bigcirc, \square$ , and  $U$  operators we compute the values assigned by the semantics

of subformulas of  $\varphi$  on suffixes of  $w$ . It is our aim to define  $\mathcal{A}_\varphi$  in a way that it simulates the above induction. For this we define the states of the automaton as sets of formulas, and every non-empty state will contain a maximal (according to subformulas relation) formula. The weights of the transitions are defined so that successful paths with non-empty states will simulate the inductive computation of the semantics of the maximal formula of the first state of the path. We consider as non-empty initial states of the automaton the ones with maximal formula  $\varphi$ . We recall from [30] that if  $\varphi \in LTL(K, AP)$ , the closure  $cl(\varphi)$  of  $\varphi$  is the smallest set  $C$  such that (a)  $\varphi \in C$ , (b) if  $\psi \wedge \xi \in C$ , or  $\psi \vee \xi \in C$ , or  $\psi U \xi \in C$ , then  $\psi, \xi \in C$ , and (c) if  $\bigcirc\psi \in C$  or  $\square\psi \in C$ , then  $\psi \in C$ . In fact  $cl(\varphi)$  contains  $\varphi$  and all its subformulas.

**Definition 6** [21] *Let  $\varphi \in LTL(K, AP)$ . A subset  $B$  of  $cl(\varphi)$  will be called  $\varphi$ -consistent if  $B = \emptyset$ , or the following conditions hold.*

- For every  $a \in AP$ ,  $a \in B$  implies  $\neg a \notin B$ ,
- $\varphi \in B$ ,
- If  $\psi \wedge \xi \in B$ , then  $\psi, \xi \in B$ ,
- If  $\psi \vee \xi \in B$  or  $\psi U \xi \in B$ , then  $\psi \in B$  or  $\xi \in B$ ,
- If  $\square\psi \in B$ , then  $\psi \in B$ .

**Example 3** *Let  $\mathbb{R}_{\min} = (\mathbb{R}_+ \cup \{\infty\}, \min, +, \infty, 0)$  be the tropical semiring. It is well known that the tropical semiring is idempotent totally commutative complete (see Chapter 5 in [21]). Let  $AP = \{a, b\}$ , and  $\varphi = a \vee b \in RULTL(\mathbb{R}_{\min}, AP)$ . Then,  $\{\emptyset, \{a \vee b, b\}, \{a \vee b, a\}, \{a \vee b, a, b\}\} \subset \mathcal{P}(cl(\varphi))$  is the set of all  $\varphi$ -consistent sets.*

**Example 4** *Let  $AP = \{a, b, c\}$ , and  $\varphi = (a \wedge 2) \vee (b \wedge 3)$ ,  $\psi = \varphi U (\bigcirc c) \in LTL(\mathbb{R}_{\min}, AP)$ . Then,  $B_\varphi = \{\varphi, a \wedge 2, a, 2\}$  is a  $\varphi$ -consistent set, and  $B_\psi = \{\psi, \varphi, a \wedge 2, a, 2\} = \{\psi\} \cup B_\varphi$  is a  $\psi$ -consistent set.*

Let  $\varphi \in LTL(K, AP)$  and  $B \neq \emptyset$  be a  $\varphi$ -consistent set. Let also  $\varphi' \in B$ , and  $\Delta, \Gamma$  be  $\varphi'$ -consistent subsets of  $B$ . Then, with standard arguments we get that  $\Delta \cup \Gamma$  is also a  $\varphi'$ -consistent subset of  $B$ . This implies that for every  $\varphi \in LTL(K, AP)$ ,  $\varphi$ -consistent set  $B$ , and  $\psi \in cl(\varphi)$ , there exists the greatest (according to subset relation)  $\psi$ -consistent subset of  $B$ . Keeping the notations of [20], we denote this set by  $M_{B, \psi}$ . Clearly, if  $\psi \in cl(\varphi) \setminus B$ , then  $M_{B, \psi} = \emptyset$ . Moreover, we shall denote a  $\varphi$ -consistent set  $B$  by  $B_\varphi$ .



**Definition 7** [21] *Let  $\varphi \in LTL(K, AP)$  and  $B_\varphi$  be a  $\varphi$ -consistent set. The finite set of formulas  $next(B_\varphi) \subseteq LTL(K, AP)$  is defined in the following way. We set  $next(\emptyset) = \{\mathbf{0}\}$  and for  $B_\varphi \neq \emptyset$ ,*

- if  $\varphi = a$ , or  $\varphi = \neg a$ , or  $\varphi = k$ ,  $a \in AP$ ,  $k \in K$ , then  $next(B_\varphi) = \{true\}$ ,
- if  $\varphi = \psi \wedge \xi$ , then  
 $next(B_\varphi) = \{\psi' \wedge \xi' \mid \psi' \in next(M_{B_\varphi, \psi}), \xi' \in next(M_{B_\varphi, \xi})\}$ ,
- if  $\varphi = \psi \vee \xi$ , then  $next(B_\varphi) = next(M_{B_\varphi, \psi}) \cup next(M_{B_\varphi, \xi})$ ,
- if  $\varphi = \bigcirc \psi$ , then  $next(B_\varphi) = \{\psi\}$ ,
- if  $\varphi = \psi U \xi$ , then  
 $next(B_\varphi) = next(M_{B_\varphi, \xi}) \cup \{\varphi \wedge \psi' \mid \psi' \in next(M_{B_\varphi, \psi})\}$ ,
- if  $\varphi = \Box \psi$ , then  $next(B_\varphi) = \{\varphi \wedge \psi' \mid \psi' \in next(M_{B_\varphi, \psi})\}$ .

The elements of  $next(B_\varphi)$  will be called *next formulas* of  $B_\varphi$ . Clearly, every formula in  $next(B_\varphi)$  is a finite conjunction of the form  $\bigwedge_{1 \leq i \leq k} \psi_i$  where for every  $1 \leq i \leq k$ ,  $\psi_i \in cl(\varphi)$  or  $\psi_i = \mathbf{0}$  or  $\psi_i = true$ . Using induction on the structure of  $\varphi$  we can easily derive that for every  $\varphi \in bLTL(K, AP)$  (resp.  $\varphi \in stLTL(K, AP)$ ,  $\varphi \in RuLTL(K, AP)$ ) and  $\varphi$ -consistent set  $B_\varphi$ , it holds  $next(B_\varphi) \subseteq bLTL(K, AP)$  (resp.  $next(B_\varphi) \subseteq stLTL(K, AP)$ ,  $next(B_\varphi) \subseteq RuLTL(K, AP)$ ). We recall that the value assigned by  $\|\varphi\|$  to an infinite word  $w$  is computed by induction on the structure of  $\varphi$ . Moreover, in the induction of the semantics of the  $\bigcirc$ ,  $\Box$ , and  $U$  operators we compute the values assigned by the semantics of subformulas of  $\varphi$  on suffixes of  $w$ . Next formulas of a  $\varphi$ -consistent set indicate the formulas whose semantics should assign a value to  $w_{\geq 1}$  so that  $(\|\varphi\|, w)$  can be effectively computed.

Next, we define inductively for every formula  $\varphi \in LTL(K, AP)$  and every  $\varphi$ -consistent set  $B_\varphi$ , a mapping  $v_{B_\varphi} : next(B_\varphi) \rightarrow K$  assigning values from  $K$  to next formulas of  $B_\varphi$ . We let  $v_{\emptyset}(\mathbf{0}) = \mathbf{0}$ . Now, assume that  $B_\varphi \neq \emptyset$ .

- For  $\varphi = k \in K$ ,  $v_{\{\varphi\}}(true) = k$ .
- For  $\varphi = a$  or  $\varphi = \neg a$  with  $a \in AP$ , we set  $v_{\{\varphi\}}(true) = \mathbf{1}$ .
- Let  $\varphi = \psi \wedge \xi$ . We let  $v_{B_\varphi}(\psi' \wedge \xi') = v_{M_{B_\varphi, \psi}}(\psi') \cdot v_{M_{B_\varphi, \xi}}(\xi')$  where  $\psi' \in next(M_{B_\varphi, \psi})$ ,  $\xi' \in next(M_{B_\varphi, \xi})$ .

- Next, let  $\varphi = \psi \vee \xi$ . For every  $\varphi' \in \text{next}(M_{B_\varphi, \psi}) \cup \text{next}(M_{B_\varphi, \xi})$  we let  $v_{B_\varphi}(\varphi') = v_{M_{B_\varphi, \psi}}(\varphi') + v_{M_{B_\varphi, \xi}}(\varphi')$  where abusing notations  $v_{M_{B_\varphi, \psi}}(\varphi')$  (resp.  $v_{M_{B_\varphi, \xi}}(\varphi')$ ) will stand for  $\mathbf{0}$  whenever  $\varphi' \notin \text{next}(M_{B_\varphi, \psi})$  (resp.  $\varphi' \notin \text{next}(M_{B_\varphi, \xi})$ ).
- Assume that  $\varphi = \bigcirc\psi$ . Then, for the unique element  $\psi$  of  $\text{next}(B_\varphi)$  we set  $v_{B_\varphi}(\psi) = \mathbf{1}$ .
- For  $\varphi = \psi U \xi$ , we let  $v_{B_\varphi}(\varphi \wedge \psi') = v_{M_{B_\varphi, \psi}}(\psi')$  where  $\psi' \in \text{next}(M_{B_\varphi, \psi})$ , and  $v_{B_\varphi}(\xi') = v_{M_{B_\varphi, \xi}}(\xi')$  with  $\xi' \in \text{next}(M_{B_\varphi, \xi})$ .
- For  $\varphi = \square\psi$ , we set  $v_{B_\varphi}(\varphi \wedge \psi') = v_{M_{B_\varphi, \psi}}(\psi')$  where  $\psi' \in \text{next}(M_{B_\varphi, \psi})$ .

**Example 5** Let  $AP = \{a, b, c\}$  and  $\varphi = (a \wedge 2) \vee (b \wedge 3) \in LTL(\mathbb{R}_{min}, AP)$ . For  $B_\varphi = \{\varphi, a \wedge 2, b \wedge 3, a, 2, b, 3\}$  we have  $\text{next}(B_\varphi) = \{\text{true} \wedge \text{true}\}$  and  $v_{B_\varphi}(\text{true} \wedge \text{true}) = 0$ .

Now, let  $\psi = \varphi U (\bigcirc c)$ . For  $B_\psi = \{\psi, \varphi, a \wedge 2, b \wedge 3, a, 2, b, 3, \bigcirc c\}$  we have  $\text{next}(B_\psi) = \{\psi \wedge (\text{true} \wedge \text{true}), c\}$ . Thus,  $v_{B_\psi}(\psi \wedge (\text{true} \wedge \text{true})) = 0$  and  $v_{B_\psi}(c) = 0$ .

In the sequel, we use next formulas of a set  $B_\varphi$  and the mapping  $v_{B_\varphi}$  to define the non- $\varepsilon$ -transitions of the desired automaton and their weights. More precisely, the states of the automaton will be consistent sets of formulas. We allow non- $\varepsilon$ -transitions with weight  $\neq \mathbf{0}$  only from a set  $B_\varphi$  to a set  $B_{\varphi'}$  with  $\varphi' \in \text{next}(B_\varphi)$ , and the weight of this transition will be equal to  $v_{B_\varphi}(\varphi')$ . We will also use  $\varepsilon$ -transitions with weight  $= \mathbf{1}$  to move from a  $\varphi$ -consistent set to a  $\varphi_{re}$ -consistent set, i.e., through the  $\varepsilon$ -transitions we will reduce formulas by erasing from conjunctions multiple copies of identical boolean formulas and the formula true. This reduction ensures that the state set of the automaton is finite. It is also crucial that we reduce formulas only with  $\varepsilon$ -transitions, since otherwise the reduction should change the structure of a formula, and thus the computation of  $v_{B_\varphi}$  would not be well defined.

Let  $\varphi, \psi \in LTL(K, AP)$  and  $B_\varphi, B_\psi$ ,  $\varphi$ -consistent,  $\psi$ -consistent sets respectively. We say that  $B_\psi$  is reachable by  $B_\varphi$ , if there exists a sequence  $\varphi_0, \dots, \varphi_j \in LTL(K, AP)$ , and  $B_{\varphi_0}, \dots, B_{\varphi_j}$ ,  $\varphi_0$ -consistent,  $\dots, \varphi_j$ -consistent set respectively, such that (i)  $\varphi_0 = \varphi, \dots, \varphi_j = \psi$ , and (ii) for every  $0 \leq l \leq j - 1$  if  $\varphi_l$  is reduced, then  $\varphi_{l+1} \in \text{next}(B_{\varphi_l})$ , otherwise  $\varphi_{l+1} = (\varphi_l)_{re}$ . Observe that, since  $\varphi_0$  is reduced, the formulas  $\varphi_l$  ( $1 \leq l \leq j$ ) satisfy condition (b) in the definition of reduced formulas. This implies that reduction, whenever it is applied, reduces only conjunction. Let  $\text{reach}(B_{\varphi_0})$

contain all sets of formulas reachable by  $B_{\varphi_0}$ . As the following remark shows  $\text{reach}(B_{\varphi_0})$  is not finite in general.

**Remark 5** [21] *We let  $AP = \{a, b, c\}$ ,  $\varphi = \Box(\Box(a \wedge 2)) \in LTL(\mathbb{R}_{\min}, AP)$ , and  $B_{\varphi} = \{\varphi, \Box(a \wedge 2), a \wedge 2, a, 2\}$ . Then, for every  $j \geq 1$ , every consistent set of the formula  $\varphi \wedge \left( \bigwedge_{1 \leq i \leq j} \psi_i \right)$  with  $\psi_i = \Box(a \wedge 2)$  ( $1 \leq i \leq j$ ) belongs to the set  $\text{reach}(B_{\varphi})$ , and hence  $\text{reach}(B_{\varphi})$  is not finite.*

*Let now  $\varphi = ((a \wedge 2)Uc)Ud$  and  $B_{\varphi} = \{\varphi, (a \wedge 2)Uc, a \wedge 2, a, 2\}$ .*

*Then, for every  $j \geq 1$ , every consistent set of the formula  $\varphi \wedge \left( \bigwedge_{1 \leq i \leq j} \psi_i \right)$  with  $\psi_i = (a \wedge 2)Uc$  ( $1 \leq i \leq j$ ) belongs to the set  $\text{reach}(B_{\varphi})$ , and hence  $\text{reach}(B_{\varphi})$  is not finite.*

However, the situation is different, if we consider formulas from  $RULTL(K, AP)$ .

**Lemma 6** *Let  $\varphi \in RULTL(K, AP)$  be reduced and  $B_{\varphi}$  be a  $\varphi$ -consistent set. Then,  $\text{reach}(B_{\varphi})$  is finite and effectively computable.*

The previous lemma is proved with the same arguments as in Lemma 94 in [21].

**Definition 8** *Let  $\varphi, \psi \in RULTL(K, AP)$ . For every  $\pi \in \mathcal{P}(AP)$  the triple  $(B_{\varphi}, \pi, B_{\psi})$  is called a next transition if the following conditions hold.*

- *For every  $a \in AP$ ,  $a \in B_{\varphi}$  implies  $a \in \pi$ , and  $\neg a \in B_{\varphi}$  implies  $a \notin \pi$ ,*
- *$\varphi$  is reduced and  $\psi \in \text{next}(B_{\varphi})$ .*

**Definition 9** *Let  $\varphi \in RULTL(K, AP)$ . Then, for every  $B_{\varphi}, B_{\varphi_{re}}$  with  $B_{\varphi} \neq \emptyset$  and  $B_{\varphi_{re}} \neq \emptyset$  the triple  $(B_{\varphi}, \varepsilon, B_{\varphi_{re}})$  is called an  $\varepsilon$ -reduction transition.*

Sometimes in the sequel an  $\varepsilon$ -reduction transition will be called for simplicity an  $\varepsilon$ -transition. Next, for every reduced  $RULTL(K, AP)$ -formula  $\varphi$  we construct an  $\varepsilon$ -wgBa  $\mathcal{A}_{\varphi}$  and show that  $\varphi$  and  $\mathcal{A}_{\varphi}$  are expressively equivalent.

**Definition 10** *Let  $\varphi \in RULTL(K, AP)$  be reduced. We define the  $\varepsilon$ -wgBa  $\mathcal{A}_{\varphi} = (Q, wt, I, \mathcal{F})$  over  $\mathcal{P}(AP)$  and  $K$  as follows. We set*

- $Q = \bigcup_{B_\varphi} (\{B_\varphi\} \cup reach(B_\varphi)),$
  - $I = \{B_\varphi \mid B_\varphi : \varphi\text{-consistent set}\},$
  - $wt(B_\psi, b, B_\xi) = \begin{cases} v_{B_\psi}(\xi) & \text{if } (B_\psi, b, B_\xi) \text{ is a next transition} \\ & \text{and } B_\xi \neq \emptyset \\ \mathbf{1} & \text{if } (B_\psi, b, B_\xi) \text{ is an } \varepsilon\text{-reduction transition} \\ \mathbf{0} & \text{otherwise} \end{cases}$
- for every  $(B_\psi, b, B_\xi) \in Q \times (\mathcal{P}(AP) \cup \{\varepsilon\}) \times Q,$  and
- $\mathcal{F} = \{F_{\varphi'U\varphi''} \mid \varphi'U\varphi'' \in cl(\varphi)\}$  where
- $$F_{\varphi'U\varphi''} = \left\{ B_{\bar{\varphi}} \in Q \mid \begin{array}{l} B_{\bar{\varphi}} \neq \emptyset, \bar{\varphi} = \bigwedge_{1 \leq i \leq k} \varphi_i \text{ is of form } A \text{ with} \\ \varphi_i \neq \varphi'U\varphi'', 1 \leq i \leq k \end{array} \right\}$$
- for every  $\varphi'U\varphi'' \in cl(\varphi).$

Observe that for every  $\varphi'U\varphi'' \in cl(\varphi),$  and every non-empty  $B_\psi, B_{\psi_{re}} \in Q,$  the relation  $B_\psi \in F_{\varphi'U\varphi''}$  implies that  $B_{\psi_{re}} \in F_{\varphi'U\varphi''},$  and vice-versa. Thus, the  $\varepsilon$ -transitions of the automaton are well defined. We note that if  $\varphi$  contains no  $U$  operators, then we have no acceptance conditions, which means that all infinite paths that start with a  $\varphi$ -consistent set are successful. Now, let  $w \in (\mathcal{P}(AP))^\omega$  and  $P_w = t_0t_1t_2 \dots$  be a successful path of  $\mathcal{A}_\varphi$  over  $w.$  If there is an  $i \geq 0,$  such that  $t_i$  is not a next transition or an  $\varepsilon$ -reduction transition, then  $weight_{\mathcal{A}_\varphi}(P_w) = \mathbf{0}.$  We shall denote by  $next_{\mathcal{A}_\varphi}(w)$  the set of all successful paths of  $\mathcal{A}_\varphi$  over  $w$  composed of next and  $\varepsilon$ -transitions only. For the rest of this section we let  $pri_{\mathcal{A}_\varphi}(w) = \{weight_{\mathcal{A}_\varphi}(P_w) \mid P_w \in next_{\mathcal{A}_\varphi}(w)\}.$

**Remark 6** Let  $\varphi \in RULTL(K, AP)$  be reduced and  $\varphi_1U\varphi_2 \in cl(\varphi)$  such that  $\varphi_1U\varphi_2$  does not appear in the scope of an always operator in  $\varphi.$  Then, for every  $w \in (\mathcal{P}(AP))^\omega$  and  $P_w \in next_{\mathcal{A}_\varphi}(w),$  there is a state in  $P_w$  such that all the subsequent states are in  $F_{\varphi_1U\varphi_2}.$  More precisely, for every  $\varphi_1U\varphi_2 \in cl(\varphi),$  if there is a next transition in the path where a next formula of the maximal  $\varphi_1U\varphi_2$ -consistent subset of the beginning state (of the transition) appears as a part of the conjunction defining the maximal formula of the arriving state, then, after a finite number of next transitions and since  $P_w$  is successful, there is a next transition where the next formula of the maximal  $\varphi_1U\varphi_2$ -consistent maximal subset of the beginning state of the transition is a next formula of the maximal  $\varphi_2$ -subset of the beginning

state, i.e., a formula that is a conjunction not containing  $\varphi_1 U \varphi_2$ . Since this holds for every appearance of  $\varphi_1 U \varphi_2$ , we conclude that there is a state in  $P_w$  such that all the subsequent states are in  $F_{\varphi_1 U \varphi_2}$ .

Next, we prove by induction on the structure of a reduced *RULTL*  $(K, AP)$ -formula  $\varphi$  that  $\mathcal{A}_\varphi$  accepts  $\|\varphi\|$ .

**Lemma 7** *Let  $\varphi = a, \varphi = \neg a$ , or  $\varphi = k$ , with  $a \in AP, k \in K$ . Then,  $\|\mathcal{A}_\varphi\| = \|\varphi\|$ .*

**Proof:** Let  $\varphi = k \in K \setminus \{\mathbf{0}, \mathbf{1}\}$ . Then, the automaton  $\mathcal{A}_\varphi = (Q, wt, I, \mathcal{F})$  is defined in the following way.

- $Q = \{\emptyset, \{k\}, \{true\}, \{\mathbf{0}\}\}$
- $wt(q, b, q') = \begin{cases} k & \text{if } q = \{k\}, q' = \{true\}, \text{ and } b \in \mathcal{P}(AP) \\ \mathbf{1} & \text{if } q = q', q \neq \emptyset, \text{ and } b = \varepsilon, \text{ or} \\ & \text{if } q = q' = true, \text{ and } b \in \mathcal{P}(AP) \\ \mathbf{0} & \text{otherwise} \end{cases}$
- $I = \{\{k\}, \emptyset\}$

The automaton contains no final subsets, i.e.,  $\mathcal{F} = \emptyset$ . Let  $P_w \in next_{\mathcal{A}_\varphi}(w)$ . Then, the next transitions appearing in the path either form the sequence

$$(\{k\}, w(0), \{true\}) ((\{true\}, w(i), \{true\}))_{i \geq 1},$$

or the sequence

$$(\emptyset, w(0), \{\mathbf{0}\}) (\{\mathbf{0}\}, w(1), \{true\}) ((\{true\}, w(i), \{true\}))_{i \geq 2}.$$

In the first case, using Property 3 we get  $weight_{\mathcal{A}_\varphi}(P_w) = Val^\omega(k, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots) = k$ , and in the latter case  $weight_{\mathcal{A}_\varphi}(P_w) = Val^\omega(\mathbf{0}, \mathbf{0}, \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots) = \mathbf{0}$ . Hence, we get for every  $w \in (\mathcal{P}(AP))^\omega$

$$(\|\mathcal{A}_\varphi\|, w) = \sum_{k' \in pri_{\mathcal{A}_\varphi}(w)} k' = k = (\|k\|, w)$$

as wanted. The lemma's claim for  $\varphi = a, \varphi = \neg a, \varphi = \mathbf{0}, \varphi = \mathbf{1}$  can be proved with similar arguments.  $\square$

**Lemma 8** *Let  $\psi, \xi \in RULTL(K, AP)$  and  $\varphi = \psi \vee \xi$ . If  $\mathcal{A}_\psi, \mathcal{A}_\xi$  recognize  $\|\psi\|, \|\xi\|$  respectively, then  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** Let  $w = \pi_0\pi_1\pi_2\dots \in (\mathcal{P}(AP))^\omega$  and  $\mathcal{A}_\psi = (Q_1, wt_1, I_1, \mathcal{F}_1)$ ,  $\mathcal{A}_\xi = (Q_2, wt_2, I_2, \mathcal{F}_2)$  and  $\mathcal{A}_\varphi = (Q, wt, I, \mathcal{F})$ . First, we prove that

$$\sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k \leq \sum_{k \in \text{pri}_{\mathcal{A}_\psi}(w)} k + \sum_{k \in \text{pri}_{\mathcal{A}_\xi}(w)} k.$$

To this end we show that for every path  $P_w \in \text{next}_{\mathcal{A}_\varphi}(w)$  there exist paths  $P_w^1 \in \text{next}_{\mathcal{A}_\psi}(w)$ ,  $P_w^2 \in \text{next}_{\mathcal{A}_\xi}(w)$  with

$$\text{weight}_{\mathcal{A}_\varphi}(P_w) = \text{weight}_{\mathcal{A}_\psi}(P_w^1) + \text{weight}_{\mathcal{A}_\xi}(P_w^2).$$

We let

$$P_w : B_\varphi \xrightarrow{*} B'_\varphi \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$

be a path in  $\text{next}_{\mathcal{A}_\varphi}(w)$  with  $\text{weight}_{\mathcal{A}_\varphi}(P_w) \neq \mathbf{0}$ . This implies that

$$wt(B'_\varphi, \pi_0, B_{\varphi^1}) \neq \mathbf{0} \text{ and } wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}) \neq \mathbf{0}$$

for every  $i \geq 1$ . Then, by definition

$$(a) \quad \varphi^1 \in \text{next}(M_{B'_\varphi, \psi}) \setminus \text{next}(M_{B'_\varphi, \xi}) \text{ and} \\ wt(B'_\varphi, \pi_0, B_{\varphi^1}) = v_{M_{B'_\varphi, \psi}}(\varphi^1),$$

or

$$(b) \quad \varphi^1 \in \text{next}(M_{B'_\varphi, \xi}) \setminus \text{next}(M_{B'_\varphi, \psi}) \text{ and} \\ wt(B'_\varphi, \pi_0, B_{\varphi^1}) = v_{M_{B'_\varphi, \xi}}(\varphi^1),$$

or

$$(c) \quad \varphi^1 \in \text{next}(M_{B'_\varphi, \psi}) \cap \text{next}(M_{B'_\varphi, \xi}) \text{ and} \\ wt(B'_\varphi, \pi_0, B_{\varphi^1}) = v_{M_{B'_\varphi, \psi}}(\varphi^1) + v_{M_{B'_\varphi, \xi}}(\varphi^1).$$

If (a) holds, then the path

$$P_w^1 : M_{B'_\varphi, \psi} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$

of  $\mathcal{A}_\psi$  over  $w$  is successful, hence  $P_w^1 \in \text{next}_{\mathcal{A}_\psi}(w)$  and  $\text{weight}_{\mathcal{A}_\varphi}(P_w) = \text{weight}_{\mathcal{A}_\psi}(P_w^1)$ , which implies that

$$\text{weight}_{\mathcal{A}_\varphi}(P_w) = \text{weight}_{\mathcal{A}_\psi}(P_w^1) + \text{weight}_{\mathcal{A}_\xi}(P_w^2)$$

for

$$P_w^2 : (\emptyset, \pi_0, \{\mathbf{0}\}) (\{\mathbf{0}\}, \pi_1, \{true\}) ((\{true\}, \pi_i, \{true\}))_{i \geq 0}.$$

If case (b) holds, then the path

$$P_w^2 : M_{B'_{\varphi}, \xi} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{r_e}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{r_e}^2} \dots$$

of  $\mathcal{A}_\xi$  over  $w$  is successful, i.e.,  $P_w^2 \in next_{\mathcal{A}_\xi}(w)$ , and

$$weight_{\mathcal{A}_\varphi}(P_w) = weight_{\mathcal{A}_\psi}(P_w^1) + weight_{\mathcal{A}_\xi}(P_w^2)$$

for

$$P_w^1 : (\emptyset, \pi_0, \{\mathbf{0}\}) (\{\mathbf{0}\}, \pi_1, \{true\}) ((\{true\}, \pi_i, \{true\}))_{i \geq 0}.$$

If case (c) holds, then

$$v_{M_{B'_{\varphi}, \psi}}(\varphi^1) + v_{M_{B'_{\varphi}, \xi}}(\varphi^1) = v_{B'_{\varphi}}(\varphi^1)$$

and for the paths  $P_w^1, P_w^2$  of  $\mathcal{A}_\psi, \mathcal{A}_\xi$  respectively, defined as in cases (a) and (b) respectively, we get

$$weight_{\mathcal{A}_\varphi}(P_w) = weight_{\mathcal{A}_\psi}(P_w^1) + weight_{\mathcal{A}_\xi}(P_w^2).$$

More precisely, for  $k_1 = v_{M_{B'_{\varphi}, \psi}}(\varphi^1), k_2 = v_{M_{B'_{\varphi}, \xi}}(\varphi^1)$  it holds

$$\begin{aligned} & weight_{\mathcal{A}_\varphi}(P_w) \\ &= Val^\omega (wt(B'_{\varphi}, \pi_0, B_{\varphi^1}), wt(B_{\varphi_{r_e}^1}, \pi_1, B_{\varphi^2}), wt(B_{\varphi_{r_e}^2}, \pi_2, B_{\varphi^3}), \dots) \\ &= Val^\omega (k_1 + k_2, wt(B_{\varphi_{r_e}^1}, \pi_1, B_{\varphi^2}), wt(B_{\varphi_{r_e}^2}, \pi_2, B_{\varphi^3}), \dots) \\ &= Val^\omega (k_1, wt(B_{\varphi_{r_e}^1}, \pi_1, B_{\varphi^2}), wt(B_{\varphi_{r_e}^2}, \pi_2, B_{\varphi^3}), \dots) \\ &+ Val^\omega (k_2, wt(B_{\varphi_{r_e}^1}, \pi_1, B_{\varphi^2}), wt(B_{\varphi_{r_e}^2}, \pi_2, B_{\varphi^3}), \dots) \\ &= weight_{\mathcal{A}_\psi}(P_w) + weight_{\mathcal{A}_\xi}(P_w), \end{aligned}$$

where the third equality holds by the distributivity of  $Val^\omega$  over finite sums.

We have thus shown that for every  $k \in pri_{\mathcal{A}_\varphi}(w)$  there exist  $k' \in pri_{\mathcal{A}_\psi}(w) + pri_{\mathcal{A}_\xi}(w)$  such that  $k \leq k'$  and by Lemmas liv, 2 we get

$$\begin{aligned} \sum_{k \in pri_{\mathcal{A}_\varphi}(w)} k &\leq \sum_{k \in pri_{\mathcal{A}_\psi}(w) + pri_{\mathcal{A}_\xi}(w)} k \\ &= \sum_{k \in pri_{\mathcal{A}_\psi}(w)} k + \sum_{k \in pri_{\mathcal{A}_\xi}(w)} k. \end{aligned} \tag{4}$$

We now show that

$$\sum_{k \in \text{pri}_{\mathcal{A}_\psi}(w)} k + \sum_{k \in \text{pri}_{\mathcal{A}_\xi}(w)} k \leq \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k.$$

Assume that

$$P_w^1 : B_\psi \xrightarrow{*} B'_\psi \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{*} B_{\psi_{re}^1} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{*} B_{\psi_{re}^2} \dots$$

is a path in  $\text{next}_{\mathcal{A}_\psi}(w)$  with  $\text{weight}_{\mathcal{A}_\psi}(P_w^1) \neq \mathbf{0}$ . We set  $B'_\varphi = B'_\psi \cup \{\psi \vee \xi\}$ . Then, the path

$$P_w : B'_\varphi \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{*} B_{\psi_{re}^1} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{*} B_{\psi_{re}^2} \dots$$

is a path of  $\mathcal{A}_\varphi$  over  $w$  in  $\text{next}_{\mathcal{A}_\varphi}(w)$  and we claim that  $\text{weight}_{\mathcal{A}_\varphi}(P_w) \geq \text{weight}_{\mathcal{A}_\psi}(P_w^1)$ . It suffices to prove that

$$\text{wt}(B'_\varphi, \pi_0, B_{\psi^1}) \geq \text{wt}_1(B'_\psi, \pi_0, B_{\psi^1}),$$

then our claim is derived by Lemma 3. If  $\psi^1 \notin \text{next}(M_{B'_\varphi, \xi})$ , then the equality holds by definition. Otherwise,  $\text{wt}(B'_\varphi, \pi_0, B_{\psi^1}) = v_{B'_\varphi}(\psi^1) = v_{M_{B'_\varphi, \psi}}(\psi^1) + v_{M_{B'_\varphi, \xi}}(\psi^1) = v_{B'_\psi}(\psi^1) + v_{M_{B'_\varphi, \xi}}(\psi^1) = \text{wt}_1(B'_\psi, \pi_0, B_{\psi^1}) + v_{M_{B'_\varphi, \xi}}(\psi^1)$ , i.e.,  $\text{wt}(B'_\varphi, \pi_0, B_{\psi^1}) \geq \text{wt}_1(B'_\psi, \pi_0, B_{\psi^1})$  as wanted. For  $P_w^1 \in \text{next}_{\mathcal{A}_\psi}(w)$  with  $\text{weight}_{\mathcal{A}_\psi}(P_w^1) = \mathbf{0}$  it trivially holds  $\text{weight}_{\mathcal{A}_\varphi}(P_w) \geq \text{weight}_{\mathcal{A}_\psi}(P_w^1)$  for every  $P_w \in \text{next}_{\mathcal{A}_\varphi}(w)$ .

Similarly, for every path  $P_w^2 \in \text{next}_{\mathcal{A}_\xi}(w)$  there exist a path  $P_w \in \text{next}_{\mathcal{A}_\varphi}(P_w)$  with  $\text{weight}_{\mathcal{A}_\varphi}(P_w) \geq \text{weight}_{\mathcal{A}_\xi}(P_w^2)$ . Hence, for every  $k \in \text{pri}_{\mathcal{A}_\psi}(P_w)$  (resp.  $k \in \text{pri}_{\mathcal{A}_\xi}(w)$ ) there exists a  $k' \in \text{pri}_{\mathcal{A}_\varphi}(w)$  such that  $k \leq k'$ . This implies that  $\sum_{k \in \text{pri}_{\mathcal{A}_\psi}(w)} k \leq \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k$  and  $\sum_{k \in \text{pri}_{\mathcal{A}_\xi}(w)} k \leq$

$\sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k$ . Again by Lemma 2 and idempotency we get,

$$\sum_{k \in \text{pri}_{\mathcal{A}_\psi}(w)} k + \sum_{k \in \text{pri}_{\mathcal{A}_\xi}(w)} k \leq \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k \quad (5)$$

as wanted.



Therefore, for every  $w \in (\mathcal{P}(AP))^\omega$  we get

$$\begin{aligned}
 (\|\mathcal{A}_\varphi\|, w) &= \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k \\
 &= \sum_{k \in \text{pri}_{\mathcal{A}_\psi}(w)} k + \sum_{k \in \text{pri}_{\mathcal{A}_\xi}(w)} k \\
 &= (\|\mathcal{A}_\psi\|, w) + (\|\mathcal{A}_\xi\|, w) \\
 &= (\|\psi\|, w) + (\|\xi\|, w) \\
 &= (\|\psi \vee \xi\|, w)
 \end{aligned}$$

where the second equality holds by 4 and 5. □

**Lemma 9** *Let  $\psi \in RULTL(K, AP)$  and  $\varphi = \bigcirc\psi$ . If  $\mathcal{A}_\psi$  recognizes  $\|\psi\|$ , then  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** Let  $\mathcal{A}_\psi = (Q', wt', I', \mathcal{F}')$ ,  $\mathcal{A}_\varphi = (Q, wt, I, \mathcal{F})$ , and  $w = \pi_0 \pi_1 \dots \in (\mathcal{P}(AP))^\omega$ . We show that  $\text{pri}_{\mathcal{A}_\varphi}(w) = \text{pri}_{\mathcal{A}_\psi}(w_{\geq 1})$ . It suffices to prove that for every  $P_w \in \text{next}_{\mathcal{A}_\varphi}(w)$ , there exists a  $P'_{w_{\geq 1}} \in \text{next}_{\mathcal{A}_\psi}(w_{\geq 1})$  with  $\text{weight}_{\mathcal{A}_\varphi}(P_w) = \text{weight}_{\mathcal{A}_\psi}(P'_{w_{\geq 1}})$  and vice-versa. For the straight implication, our claim clearly holds if the empty state appears in  $P_w$ . Now, let  $P_w \in \text{next}_{\mathcal{A}_\varphi}(w)$  be a path with non-empty states that starts with a next transition, i.e., it is of the form

$$P_w : B_\varphi \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$

We have  $\varphi^1 = \psi$ , and  $wt(B_\varphi, \pi_0, B_{\varphi^1}) = v_{B_\varphi}(\psi) = \mathbf{1}$ , where the last equality holds by the definition of  $v_{B_\varphi}$ . Then, the sequence

$$P'_{w_{\geq 1}} : B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$

is a path of  $\mathcal{A}_\psi$  over  $w_{\geq 1}$  with  $P'_{w_{\geq 1}} \in \text{next}_{\mathcal{A}_\psi}(w_{\geq 1})$ , and

$$\begin{aligned}
 \text{weight}_{\mathcal{A}_\varphi}(P_w) &= \text{Val}^\omega(\mathbf{1}, wt(B_{\varphi_{re}^1}, \pi_1, B_{\varphi^2}), wt(B_{\varphi_{re}^2}, \pi_2, B_{\varphi^3}), \dots) \\
 &= \text{Val}^\omega(wt'(B_{\varphi_{re}^1}, \pi_1, B_{\varphi^2}), wt'(B_{\varphi_{re}^2}, \pi_2, B_{\varphi^3}), \dots) \\
 &= \text{weight}_{\mathcal{A}_\psi}(P'_{w_{\geq 1}}).
 \end{aligned}$$

The second equality holds since  $wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}) = wt'(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}})$  for every  $i \geq 1$ , and by Property 2.

Conversely, let  $P'_{w_{\geq 1}} : B_{\psi} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{*} B_{\psi_{re}^2} \dots$  be a path in  $next_{\mathcal{A}_{\psi}}(w_{\geq 1})$  with non-empty states. Then, the sequence

$$P_w : B_{\psi} \cup \{\circ\psi\} \xrightarrow{\pi_0} B_{\psi} \xrightarrow{\pi_1} B_{\psi^1} \xrightarrow{*} B_{\psi_{re}^1} \dots$$

is a path of  $\mathcal{A}_{\varphi}$  over  $w$  with  $P_w \in next_{\mathcal{A}_{\varphi}}(w)$ , and

$$weight_{\mathcal{A}_{\varphi}}(P_w) = weight_{\mathcal{A}_{\psi}}(P'_{w_{\geq 1}}).$$

If the empty state occurs in  $P'_{w_{\geq 1}}$  it is obvious.

Thus, for every  $w \in (\mathcal{P}(AP))^{\omega}$  we have

$$(\|\mathcal{A}_{\varphi}\|, w) = \sum_{k \in pri_{\mathcal{A}_{\varphi}}(w)} k = \sum_{k \in pri_{\mathcal{A}_{\psi}}(w_{\geq 1})} k = (\|\psi\|, w_{\geq 1}) = (\|\circ\psi\|, w),$$

as wanted.  $\square$

For every  $\varphi \in LTL(K, AP)$ , and every  $B_{\varphi}$ , we let  $\widehat{next}(B_{\varphi})$  be the subset of  $next(B_{\varphi})$  containing all  $\bigwedge_{1 \leq i \leq k} \psi_i \in next(B_{\varphi})$  of Form A where  $\psi_i \neq \mathbf{0}$  for all  $i \in \{1, \dots, k\}$ . The subsequent three lemmas will contribute to the proof of the remaining induction steps.

**Lemma 10** *Let  $\varphi \in RULTL(K, AP)$ , and  $B_{\varphi}, B'_{\varphi} \neq \emptyset$  be  $\varphi$ -consistent sets with  $B_{\varphi} \subseteq B'_{\varphi}$ . Then,  $\widehat{next}(B_{\varphi}) \subseteq next(B'_{\varphi})$  and for every  $\psi \in \widehat{next}(B_{\varphi})$  it holds  $v_{B'_{\varphi}}(\psi) \geq v_{B_{\varphi}}(\psi)$ .*

**Proof:** For atomic propositions  $a, \neg a \in AP$ , and for  $k \in K$  our claim is obvious. Let  $\varphi = \lambda \vee \xi$ . If  $M_{B_{\varphi}, \lambda} \neq \emptyset$  and  $M_{B_{\varphi}, \xi} \neq \emptyset$ , then

$$\begin{aligned} \widehat{next}(B_{\varphi}) &= \widehat{next}(M_{B_{\varphi}, \lambda}) \cup \widehat{next}(M_{B_{\varphi}, \xi}) \\ &\subseteq next(M_{B'_{\varphi}, \lambda}) \cup next(M_{B'_{\varphi}, \xi}) \\ &= next(B'_{\varphi}) \end{aligned}$$

where the inclusion holds by the induction hypothesis since  $M_{B_{\varphi}, \lambda} \subseteq M_{B'_{\varphi}, \lambda}$  and these two sets are  $\lambda$ -consistent, and similarly  $M_{B_{\varphi}, \xi} \subseteq M_{B'_{\varphi}, \xi}$  and these two sets are  $\xi$ -consistent.

Moreover, for every  $\psi \in \widehat{next}(M_{B_{\varphi}, \lambda}) \cap \widehat{next}(M_{B_{\varphi}, \xi})$ , we have

$$\begin{aligned} v_{B'_{\varphi}}(\psi) &= v_{M_{B'_{\varphi}, \lambda}}(\psi) + v_{M_{B'_{\varphi}, \xi}}(\psi) \\ &\geq v_{M_{B_{\varphi}, \lambda}}(\psi) + v_{M_{B_{\varphi}, \xi}}(\psi) \\ &= v_{B_{\varphi}}(\psi), \end{aligned}$$

whereas for  $\psi \in \widehat{next}(M_{B_\varphi, \lambda}) \setminus \widehat{next}(M_{B_\varphi, \xi})$  we have

$$v_{B'_\varphi}(\psi) \geq v_{M_{B'_\varphi, \lambda}}(\psi) \geq v_{M_{B_\varphi, \lambda}}(\psi) = v_{B_\varphi}(\psi).$$

In the same way, for  $\psi \in \widehat{next}(M_{B_\varphi, \xi}) \setminus \widehat{next}(M_{B_\varphi, \lambda})$  we get  $v_{B'_\varphi}(\psi) \geq v_{B_\varphi}(\psi)$ . Now, if  $M_{B_\varphi, \lambda} \neq \emptyset$  and  $M_{B_\varphi, \xi} = \emptyset$ , then

$$\widehat{next}(B_\varphi) = \widehat{next}(M_{B_\varphi, \lambda}) \subseteq next(M_{B'_\varphi, \lambda}) \subseteq next(B'_\varphi)$$

and for every  $\psi \in \widehat{next}(B_\varphi)$ , we have

$$v_{B'_\varphi}(\psi) \geq v_{M_{B'_\varphi, \lambda}}(\psi) \geq v_{M_{B_\varphi, \lambda}}(\psi) = v_{B_\varphi}(\psi).$$

The case  $M_{B_\varphi, \lambda} = \emptyset$  and  $M_{B_\varphi, \xi} \neq \emptyset$  is treated similarly.

Let  $\varphi = \lambda \wedge \xi$  such that  $\lambda \in RULTL(K, AP)$ ,  $\xi \in bLTL(K, AP)$ .

Then,

$$\begin{aligned} \widehat{next}(B_\varphi) &= \left\{ \lambda' \wedge \xi' \mid \lambda' \in \widehat{next}(M_{B_\varphi, \lambda}), \xi' \in \widehat{next}(M_{B_\varphi, \xi}) \right\} \\ &\subseteq \left\{ \lambda' \wedge \xi' \mid \lambda' \in next(M_{B'_\varphi, \lambda}), \xi' \in next(M_{B'_\varphi, \xi}) \right\} \\ &= next(B'_\varphi) \end{aligned}$$

where the inclusion again holds by the induction hypothesis since  $M_{B_\varphi, \lambda} \subseteq M_{B'_\varphi, \lambda}$  and  $M_{B_\varphi, \xi} \subseteq M_{B'_\varphi, \xi}$ . Moreover, for every  $\psi = \lambda' \wedge \xi'$  with  $\lambda' \in \widehat{next}(M_{B_\varphi, \lambda})$ ,  $\xi' \in \widehat{next}(M_{B_\varphi, \xi})$  we have

$$\begin{aligned} v_{B'_\varphi}(\psi) &= v_{M_{B'_\varphi, \lambda}}(\lambda') \cdot v_{M_{B'_\varphi, \xi}}(\xi') \\ &\geq v_{M_{B_\varphi, \lambda}}(\lambda') \cdot v_{M_{B_\varphi, \xi}}(\xi') \\ &= v_{B_\varphi}(\psi) \end{aligned}$$

where the inequality holds by induction hypothesis and the fact that  $\xi$  is boolean. More precisely, by induction hypothesis it holds  $v_{M_{B'_\varphi, \lambda}}(\lambda') \geq v_{M_{B_\varphi, \lambda}}(\lambda')$ , and  $v_{M_{B'_\varphi, \xi}}(\xi') \geq v_{M_{B_\varphi, \xi}}(\xi')$ , and since  $\xi$  is boolean  $v_{M_{B'_\varphi, \xi}}(\xi') = \mathbf{0}$  implies  $v_{M_{B_\varphi, \xi}}(\xi') = \mathbf{0}$ , and  $v_{M_{B_\varphi, \xi}}(\xi') = \mathbf{1}$  implies  $v_{M_{B'_\varphi, \xi}}(\xi') = \mathbf{1}$ , and thus we conclude the inequality.

Assume now that  $\varphi = \bigcirc \xi$ . Then,  $\widehat{next}(B_\varphi) = \{\xi\}$  or  $\widehat{next}(B_\varphi) = \emptyset$ . In both cases  $\widehat{next}(B_\varphi) \subseteq next(B'_\varphi) = \{\xi\}$ . In addition if  $next(B_\varphi) = \{\xi\}$ , then  $v_{B'_\varphi}(\xi) = \mathbf{1} = v_{B_\varphi}(\xi)$ .

Next, let  $\varphi = \lambda U \xi$  where  $\lambda, \xi \in stLTL(K, AP)$

$$\begin{aligned} & \widehat{next}(B_\varphi) \\ &= \left\{ \varphi \wedge \lambda' \mid \lambda' \in \widehat{next}(M_{B_\varphi, \lambda}) \right\} \\ & \cup \left\{ \xi' \mid \xi' \in \widehat{next}(M_{B_\varphi, \xi}) \right\} \\ & \subseteq \left\{ \varphi \wedge \lambda' \mid \lambda' \in next(M_{B'_\varphi, \lambda}) \right\} \\ & \cup \left\{ \xi' \mid \xi' \in next(M_{B'_\varphi, \xi}) \right\} \\ &= next(B'_\varphi). \end{aligned}$$

For  $\psi \in \widehat{next}(B_\varphi)$  with  $\psi = \varphi \wedge \lambda'$  and  $\lambda' \in \widehat{next}(M_{B_\varphi, \lambda})$ , we have  $v_{B'_\varphi}(\psi) = v_{M_{B'_\varphi, \lambda}}(\lambda') \geq v_{M_{B_\varphi, \lambda}}(\lambda') = v_{B_\varphi}(\psi)$ . For  $\psi \in \widehat{next}(B_\varphi)$  with  $\psi = \xi' \in \widehat{next}(M_{B_\varphi, \xi})$  we get  $v_{B'_\varphi}(\psi) = v_{M_{B'_\varphi, \xi}}(\xi') \geq v_{M_{B_\varphi, \xi}}(\xi') = v_{B_\varphi}(\psi)$ .

Finally, if  $\varphi = \square \xi$  where  $\xi \in stLTL(K, AP)$ , then

$$\begin{aligned} \widehat{next}(B_\varphi) &= \left\{ \varphi \wedge \xi' \mid \xi' \in \widehat{next}(M_{B_\varphi, \xi}) \right\} \\ &\subseteq \left\{ \varphi \wedge \xi' \mid \xi' \in next(M_{B'_\varphi, \xi}) \right\} \\ &= next(B'_\varphi), \end{aligned}$$

and for every  $\psi = \varphi \wedge \xi'$ , with  $\xi' \in \widehat{next}(M_{B_\varphi, \xi})$ , we have  $v_{B'_\varphi}(\psi) = v_{M_{B'_\varphi, \xi}}(\xi') \geq v_{M_{B_\varphi, \xi}}(\xi') = v_{B_\varphi}(\psi)$ .  $\square$

**Lemma 11** *Let  $\psi \in bLTL(K, AP)$  and  $\xi \in RULTL(K, AP)$ . If  $\psi, \xi$  are reduced and  $\varphi = (\psi \wedge \xi)_{re}$ , then for every  $B_\varphi \neq \emptyset$  and  $\varphi' \in next(B_\varphi)$  there exist a  $\psi$ -consistent set  $B_\psi$  and a  $\xi$ -consistent set  $B_\xi$  such that for some  $\psi' \in next(B_\psi)$ ,  $\xi' \in next(B_\xi)$  it holds  $\varphi'_{re} = (\psi'_{re} \wedge \xi'_{re})_{re}$  and  $v_{B_\varphi}(\varphi') = v_{B_\psi}(\psi') \cdot v_{B_\xi}(\xi')$ .*

**Proof:** Assume first that  $\psi \neq true$  and  $\xi \neq true$ . We point out the following cases.

- (a)  $\psi = \bigwedge_{1 \leq i \leq m_1} \psi_i$ ,  $\xi = \bigwedge_{1 \leq j \leq m_2} \xi_j$  and there exist  $i_1, \dots, i_h \in \{1, \dots, m_1\}$  and  $j_1, \dots, j_h \in \{1, \dots, m_2\}$  such that  $\psi_{i_1} = \xi_{j_1}, \dots, \psi_{i_h} = \xi_{j_h}$ . Then,

$$\varphi = \left( \bigwedge_{1 \leq i \leq m_1} \psi_i \right) \wedge \left( \bigwedge_{\substack{1 \leq j \leq m_2 \\ j \neq j_1, \dots, j \neq j_h}} \xi_j \right)$$

and

$$\varphi' = \left( \bigwedge_{1 \leq i \leq m_1} \psi'_i \right) \wedge \left( \bigwedge_{\substack{1 \leq j \leq m_2 \\ j \neq j_1, \dots, j \neq j_h}} \xi'_j \right)$$

where  $\psi'_i \in \text{next}(M_{B_\varphi, \psi_i})$ ,  $i \in \{1, \dots, m_1\}$ , and  $\xi'_j \in \text{next}(M_{B_\varphi, \xi_j})$ ,  $j \in \{1, \dots, m_2\} \setminus \{j_1, \dots, j_h\}$ . We let

$$B_\psi = \{\psi\} \cup \left( \bigcup_{1 \leq i \leq m_1} M_{B_\varphi, \psi_i} \right)$$

and

$$B_\xi = \{\xi\} \cup \left( \bigcup_{1 \leq j \leq m_2} M_{B_\varphi, \xi_j} \right).$$

Then,  $\psi' = \bigwedge_{1 \leq i \leq m_1} \psi'_i \in \text{next}(B_\psi)$  and  $\xi' = \xi'' \wedge (\psi'_{i_1} \wedge \dots \wedge \psi'_{i_h}) \in \text{next}(B_\xi)$  where  $\xi'' = \bigwedge_{\substack{1 \leq j \leq m_2 \\ j \neq j_1, \dots, j \neq j_h}} \xi'_j$ . It follows that  $\psi'_{re} = (\psi'_{re} \wedge \xi'_{re})_{re}$ , and since  $M_{B_\psi, \psi_i} = M_{B_\varphi, \psi_i}$ ,  $M_{B_\xi, \xi_j} = M_{B_\varphi, \xi_j}$  ( $1 \leq i \leq m_1, 1 \leq j \leq m_2$ ) we get that

$$\begin{aligned} v_{B_\varphi}(\varphi') &= \prod_{1 \leq i \leq m_1} v_{M_{B_\varphi, \psi_i}}(\psi'_i) \cdot \prod_{\substack{1 \leq j \leq m_2 \\ j \neq j_1, \dots, j \neq j_h}} v_{M_{B_\varphi, \xi_j}}(\xi'_j) \\ &= \prod_{1 \leq i \leq m_1} v_{M_{B_\varphi, \psi_i}}(\psi'_i) \cdot \prod_{\substack{1 \leq j \leq m_2 \\ j \neq j_1, \dots, j \neq j_h}} v_{M_{B_\xi, \xi_j}}(\xi'_j) \\ &\quad \cdot \prod_{1 \leq k \leq h} v_{M_{B_\xi, \psi_{i_k}}}(\psi'_{i_k}) \\ &= v_{B_\psi}(\psi') \cdot v_{B_\xi}(\xi'). \end{aligned}$$

(b)  $(\psi \wedge \xi)_{re} = \psi \wedge \xi$ , and our claim follows by definition.

Now, let  $\psi = \text{true}$ , and  $\xi \neq \text{true}$ . Then,  $\varphi = (\psi \wedge \xi)_{re} = \xi$ , and for  $B_\xi = B_\varphi, B_\psi = \{\text{true}\}, \xi' = \varphi'$ , and  $\psi' = \text{true}$  our claim obviously holds.

For the case where  $\psi \neq true$ , and  $\xi = true$ , and the case  $\psi = \xi = true$ , we act similarly.  $\square$

**Lemma 12** *Let  $\psi \in bLTL(K, AP)$  and  $\xi \in RULTL(K, AP)$  be reduced and  $\pi \in \mathcal{P}(AP)$ . If  $(B_\psi, \pi, B_{\psi'})$ ,  $(B_\xi, \pi, B_{\xi'})$  are next transitions with  $\psi' \in \widehat{next}(B_\psi)$ ,  $\xi' \in \widehat{next}(B_\xi)$ , and  $v_{B_\psi}(\psi') \neq \mathbf{0}$ ,  $v_{B_\xi}(\xi') \neq \mathbf{0}$ , then for  $\varphi = (\psi \wedge \xi)_{re}$  there exist  $B_\varphi \neq \emptyset$ ,  $\varphi' \in \widehat{next}(B_\varphi)$ , and  $\psi'' \in RULTL(K, AP)$  such that*

(i)  $(B_\varphi, \pi, B_{\varphi'})$  is a next transition for every  $B_{\varphi'}$  and  

$$v_{B_\varphi}(\varphi') \geq v_{B_\psi}(\psi') \cdot v_{B_\xi}(\xi'),$$

(ii)  $\varphi'_{re} = (\psi'_{re} \wedge \psi''_{re})_{re}$ , and for every infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\xi^0} \xrightarrow{\pi_0} B_{\xi^1} \xrightarrow{\varepsilon} B_{\xi^1_{re}} \xrightarrow{\pi_1} B_{\xi^2} \xrightarrow{\varepsilon} B_{\xi^2_{re}} \dots$$

with  $\xi^0 = \xi'_{re}$  and  $v_{B_{\xi^i_{re}}}(\xi^{i+1}) \neq \mathbf{0}$  ( $i \geq 0$ ), there exist an infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\lambda^0} \xrightarrow{\pi_0} B_{\lambda^1} \xrightarrow{\varepsilon} B_{\lambda^1_{re}} \xrightarrow{\pi_1} B_{\lambda^2} \xrightarrow{\varepsilon} B_{\lambda^2_{re}} \dots$$

with  $\lambda^0 = \psi''_{re}$  and  $v_{B_{\lambda^i_{re}}}(\lambda^{i+1}) = v_{B_{\xi^i_{re}}}(\xi^{i+1})$  for every  $i \geq 0$ .

**Proof:** First, we assume that both  $\psi, \xi$  are different from *true* and we point out the following cases.

(a)  $\psi = \bigwedge_{1 \leq i \leq m_1} \psi_i$ ,  $\xi = \left( \bigwedge_{1 \leq j \leq m_2+1} \xi_j \right)$  with  $\xi_{m_2+1} \in RULTL(K, AP)$ ,  $\xi_j \in bLTL(K, AP)$  for every  $j \in \{1, \dots, m_2\}$ , and there exist  $i_1, \dots, i_h \in \{1, \dots, m_1\}$ ,  $j_1, \dots, j_h \in \{1, \dots, m_2\}$  such that  $\psi_{i_1} = \xi_{j_1}, \dots, \psi_{i_h} = \xi_{j_h}$ . Then,  $\psi' = \bigwedge_{1 \leq i \leq m_1} \psi'_i$ ,  $\xi' = \bigwedge_{1 \leq j \leq m_2+1} \xi'_j$  where  $\psi'_i \in \widehat{next}(M_{B_\psi, \psi_i})$  for every  $i \in \{1, \dots, m_1\}$ ,  $\xi'_j \in \widehat{next}(M_{B_\xi, \xi_j})$  for every  $\{1, \dots, m_2+1\}$ .

Clearly,  $\varphi = \left( \bigwedge_{1 \leq i \leq m_1} \psi_i \right) \wedge \left( \bigwedge_{\substack{1 \leq j \leq m_2+1 \\ j \neq j_1, \dots, j \neq j_h}} \xi_j \right)$ . Let  $B_\varphi = \{\varphi\} \cup$

$\left( \bigcup_{1 \leq i \leq m_1} M_{B_\psi, \psi_i} \right) \cup \left( \bigcup_{1 \leq j \leq m_2+1} M_{B_\xi, \xi_j} \right)$ . We can prove that  $B_\varphi$  is a  $\varphi$ -consistent set (see proof of Lemma 107 in [21]). Moreover,  $M_{B_\psi, \psi_i} \subseteq M_{B_\varphi, \psi_i}$  and  $M_{B_\xi, \xi_j} \subseteq M_{B_\varphi, \xi_j}$  ( $1 \leq i \leq m_1, 1 \leq j \leq m_2$ ), which by Lemma 10 implies that  $\varphi' = \left( \bigwedge_{1 \leq i \leq m_1} \psi'_i \right) \wedge \left( \bigwedge_{\substack{1 \leq j \leq m_2+1 \\ j \neq j_1, \dots, j \neq j_h}} \xi'_j \right) \in \widehat{\text{next}}(B_\varphi)$ .

Therefore,  $(B_\varphi, \pi, B_{\varphi'})$  is a next transition and

$$\begin{aligned} v_{B_\varphi}(\varphi') &= \prod_{1 \leq i \leq m_1} v_{M_{B_\varphi, \psi_i}}(\psi'_i) \cdot \left( \prod_{\substack{1 \leq j \leq m_2+1 \\ j \neq j_1, \dots, j \neq j_h}} v_{M_{B_\varphi, \xi_j}}(\xi'_j) \right) \\ &\geq \prod_{1 \leq i \leq m_1} v_{M_{B_\psi, \psi_i}}(\psi'_i) \cdot \left( \prod_{\substack{1 \leq j \leq m_2+1 \\ j \neq j_1, \dots, j \neq j_h}} v_{M_{B_\xi, \xi_j}}(\xi'_j) \right) \\ &\quad \cdot \prod_{1 \leq k \leq h} v_{M_{B_\xi, \xi_{j_k}}}(\xi'_{j_k}) \\ &= v_{B_\psi}(\psi') \cdot v_{B_\xi}(\xi') \end{aligned}$$

where the inequality is concluded due to the following: By Lemma 10, we get  $v_{M_{B_\varphi, \psi_i}}(\psi'_i) \geq v_{M_{B_\psi, \psi_i}}(\psi'_i)$ , and  $v_{M_{B_\varphi, \xi_j}}(\xi'_j) \geq v_{M_{B_\xi, \xi_j}}(\xi'_j)$  for every  $1 \leq i \leq m_1, 1 \leq j \leq m_2 + 1$ . It holds  $v_{M_{B_\psi, \psi_i}}(\psi'_i) = \mathbf{1}$ ,  $v_{M_{B_\xi, \xi_j}}(\xi'_j) = \mathbf{1}$  ( $1 \leq i \leq m_1, 1 \leq j \leq m_2$ ), which implies  $v_{M_{B_\varphi, \psi_i}}(\psi'_i) = \mathbf{1}$ ,  $v_{M_{B_\varphi, \xi_j}}(\xi'_j) = \mathbf{1}$  ( $1 \leq i \leq m_1, 1 \leq j \leq m_2$ ). Then, taking into account Remark 1, we conclude the inequality.

We have completed the proof of (i). In order to prove (ii) we set

$$\psi'' = \bigwedge_{\substack{1 \leq j \leq m_2+1 \\ j \neq j_1, \dots, j \neq j_h}} \xi'_j.$$

It holds  $\varphi'_{re} = (\psi'_{re} \wedge \psi''_{re})_{re}$ . We consider now the

infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\xi^0} \xrightarrow{\pi_0} B_{\xi^1} \xrightarrow{\varepsilon} B_{\xi^1_{re}} \xrightarrow{\pi_1} B_{\xi^2} \xrightarrow{\varepsilon} B_{\xi^2_{re}} \dots$$

with  $\xi^0 = \xi'_{re}$  and  $v_{B_{\xi_{re}^i}}(\xi^{i+1}) \neq \mathbf{0}$  ( $i \geq 0$ ). Clearly,

$$\xi^0 = \xi'_{re} = \left( \psi''_{re} \wedge (\psi'_{j_1} \wedge \dots \wedge \psi'_{j_h})_{re} \right)_{re}.$$

Then, for  $\lambda^0 = \psi''_{re}$ , and  $\zeta^0 = \left( \xi'_{j_1} \wedge \dots \wedge \xi'_{j_h} \right)_{re}$ , by induction on  $i$  and Lemma 11, we obtain that for every  $i \geq 0$ , there exist a  $\lambda_{re}^i$ -consistent set  $B_{\lambda_{re}^i}$ , and a  $\zeta_{re}^i$ -consistent set  $B_{\zeta_{re}^i}$ , and formulas  $\lambda^{i+1} \in next(B_{\lambda_{re}^i})$ ,  $\zeta^{i+1} \in next(B_{\zeta_{re}^i})$  such that  $\xi_{re}^{i+1} = (\lambda_{re}^{i+1} \wedge \zeta_{re}^{i+1})_{re}$ , and  $v_{B_{\xi_{re}^i}}(\xi^{i+1}) = v_{B_{\lambda_{re}^i}}(\lambda^{i+1}) \cdot v_{B_{\zeta_{re}^i}}(\zeta^{i+1})$ . For every  $i \geq 0$ ,  $v_{B_{\xi_{re}^i}}(\xi^{i+1}) \neq \mathbf{0}$  and  $\zeta_{re}^i$  is boolean, hence  $v_{B_{\zeta_{re}^i}}(\zeta^{i+1}) = \mathbf{1}$ , i.e.,  $v_{B_{\xi_{re}^i}}(\xi^{i+1}) = v_{B_{\lambda_{re}^i}}(\lambda^{i+1})$  for every  $i \geq 0$ . So, the sequence

$$B_{\lambda^0} \xrightarrow{\pi_0} B_{\lambda^1} \xrightarrow{*} B_{\lambda_{re}^1} \xrightarrow{\pi_1} B_{\lambda^2} \xrightarrow{*} B_{\lambda_{re}^2} \dots$$

satisfies the lemma's claim.

- (b) If  $(\psi \wedge \xi)_{re} = \psi \wedge \xi$ , we set  $B_\varphi = \{\varphi\} \cup B_\psi \cup B_\xi$ , and we proceed in the same way. Finally, it is trivial to prove our claim in the cases where at least one of  $\psi, \xi$  equals to *true*.  $\square$

**Lemma 13** *Let  $\varphi = \psi \wedge \xi$  with  $\psi \in bLTL(K, AP)$  and  $\xi \in RULTL(K, AP)$ . If  $\mathcal{A}_\psi, \mathcal{A}_\xi$  recognize  $\|\psi\|, \|\xi\|$  respectively, then  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** Let  $\mathcal{A}_\psi = (Q_1, wt_1, I_1, \mathcal{F}_1)$ ,  $\mathcal{A}_\xi = (Q_2, wt_2, I_2, \mathcal{F}_2)$ ,  $\mathcal{A}_\varphi = (Q, wt, I, \mathcal{F})$ , and  $w = \pi_0 \pi_1 \pi_2 \dots \in (\mathcal{P}(AP))^\omega$ . First, we show that  $(\|\mathcal{A}_\varphi\|, w) \leq (\|\varphi\|, w)$ . In order to do this, it is necessary to prove that for every  $P_w \in next_{\mathcal{A}_\varphi}(w)$ , there exist  $P_w^1 \in next_{\mathcal{A}_\psi}(w)$  and  $P_w^2 \in next_{\mathcal{A}_\xi}(w)$  such that  $weight_{\mathcal{A}_\varphi}(P_w) = weight_{\mathcal{A}_\psi}(P_w^1) \cdot weight_{\mathcal{A}_\xi}(P_w^2)$ . If  $weight_{\mathcal{A}_\varphi}(P_w) = \mathbf{0}$ , then the paths  $P_w^1, P_w^2$  can be defined in the obvious way. Otherwise, it is possible to define the paths  $P_w^1, P_w^2$  due to the following. At every next transition of  $P_w$  the automaton  $\mathcal{A}_\varphi$  simulates two next transitions, one of  $\mathcal{A}_\psi$  and one of  $\mathcal{A}_\xi$ , and multiplies their weights. Since  $\varphi$  is reduced there are two possibilities. Either  $P_w$  starts with a next transition, or if not, before realizing the first next transition the automaton realizes a finite number of  $\varepsilon$ -transitions connecting  $\varphi$ -consistent sets. In the second case, the weight of the path coincides with the weight of the suffix path starting with the first next transition. So it suffices to prove our claim for paths  $P_w \in next_{\mathcal{A}_\varphi}(w)$  with non-zero weight of the form

$$P_w : B_{\varphi^0} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$



where  $\varphi^0 = (\psi \wedge \xi)_{re} = \psi \wedge \xi$ .

We let  $\psi^0 = \psi$  and  $\xi^0 = \xi$ . By induction and Lemma 11, we get that for every  $i \geq 0$ , there exist a  $\psi_{re}^i$ -consistent set  $B_{\psi_{re}^i}$ , a  $\xi_{re}^i$ -consistent set  $B_{\xi_{re}^i}$ , and formulas  $\psi^{i+1} \in next(B_{\psi_{re}^i})$ ,  $\xi^{i+1} \in next(B_{\xi_{re}^i})$  such that  $\varphi_{re}^{i+1} = (\psi_{re}^{i+1} \wedge \xi_{re}^{i+1})_{re}$  and  $wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}) = v_{B_{\psi_{re}^i}}(\psi^{i+1}) \cdot v_{B_{\xi_{re}^i}}(\xi^{i+1})$ .<sup>2</sup>

So, the sequences

$$P_w^1 : B_{\psi^0} \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{\varepsilon} B_{\psi_{re}^1} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{\varepsilon} B_{\psi_{re}^2} \dots$$

and

$$P_w^2 : B_{\xi^0} \xrightarrow{\pi_0} B_{\xi^1} \xrightarrow{\varepsilon} B_{\xi_{re}^1} \xrightarrow{\pi_1} B_{\xi^2} \xrightarrow{\varepsilon} B_{\xi_{re}^2} \dots$$

form successful paths of next and  $\varepsilon$ -transitions of  $\mathcal{A}_\psi$  and  $\mathcal{A}_\xi$ , respectively. We note that in the above paths for every  $i \geq 0$ ,  $B_{\psi^i}, B_{\xi^i}$  are non-empty sets and  $\psi^i$ -consistent,  $\xi^i$ -consistent respectively. It holds

$$\begin{aligned} weight_{\mathcal{A}_\varphi}(P_w) &= Val^\omega(wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}))_{i \geq 0} \\ &= Val^\omega(wt_1(B_{\psi_{re}^i}, \pi_i, B_{\psi^{i+1}}) \cdot wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi^{i+1}}))_{i \geq 0} \\ &= Val^\omega(wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi^{i+1}}))_{i \geq 0} \\ &= Val^\omega(wt_1(B_{\psi_{re}^i}, \pi_i, B_{\psi^i}))_{i \geq 0} \cdot Val^\omega(wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi^i}))_{i \geq 0} \\ &= weight_{\mathcal{A}_\psi}(P_w^1) \cdot weight_{\mathcal{A}_\xi}(P_w^2) \end{aligned}$$

where the third and fourth equality hold by the fact that  $P_w^1$  is a path on the boolean formula  $\psi$  with  $weight_{\mathcal{A}_\psi}(P_w^1) = \mathbf{1}$ , i.e., the weight of each next transition appearing in the path is equal to  $\mathbf{1}$ . We thus conclude that for every  $k \in pri_{\mathcal{A}_\varphi}(w) \setminus \{\mathbf{0}\}$  there exist  $k_1 \in pri_{\mathcal{A}_\psi}(w) \setminus \{\mathbf{0}\} = \{\mathbf{1}\}$ ,  $k_2 \in pri_{\mathcal{A}_\xi}(w) \setminus \{\mathbf{0}\}$  such that  $k \leq k_1 \cdot k_2 = k_2$  which implies

$$\begin{aligned} (\|\mathcal{A}_\varphi\|, w) &= \sum_{k \in pri_{\mathcal{A}_\varphi}(w) \setminus \{\mathbf{0}\}} k \\ &\leq \sum_{k_2 \in pri_{\mathcal{A}_\xi}(w) \setminus \{\mathbf{0}\}} k_2 \\ &= \left( \sum_{k_1 \in pri_{\mathcal{A}_\psi}(w) \setminus \{\mathbf{0}\}} k_1 \right) \cdot \left( \sum_{k_2 \in pri_{\mathcal{A}_\xi}(w) \setminus \{\mathbf{0}\}} k_2 \right) \\ &= (\|\psi\|, w) \cdot (\|\xi\|, w) \\ &= (\|\varphi\|, w). \end{aligned}$$

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<sup>2</sup>We shall call this inductive procedure, Procedure 1.

Clearly, if  $(\|\mathcal{A}_\varphi\|, w) = \mathbf{0}$ , it holds  $(\|\mathcal{A}_\varphi\|, w) \leq (\|\varphi\|, w)$ .

Now, we prove that  $(\|\varphi\|, w) \leq (\|\mathcal{A}_\varphi\|, w)$ . For this, we first prove that for every  $P_w^1 \in \text{next}_{\mathcal{A}_\psi}(w)$  with  $\text{weight}_{\mathcal{A}_\psi}(w) = \mathbf{1}$  and every  $P_w^2 \in \text{next}_{\mathcal{A}_\xi}(w)$  with  $\text{weight}_{\mathcal{A}_\xi}(w) \neq \mathbf{0}$  there is a  $P_w \in \text{next}_{\mathcal{A}_\varphi}(w)$  such that  $\text{weight}_{\mathcal{A}_\psi}(P_w^1) \cdot \text{weight}_{\mathcal{A}_\xi}(P_w^2) = \text{weight}_{\mathcal{A}_\xi}(P_w^2) \leq \text{weight}_{\mathcal{A}_\varphi}(P_w)$ . We let

$$P_w^1 : B_{\psi^0} \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{\varepsilon} B_{\psi_{re}^1} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{\varepsilon} B_{\psi_{re}^2} \dots$$

and

$$P_w^2 : B_{\xi^0} \xrightarrow{\pi_0} B_{\xi^1} \xrightarrow{\varepsilon} B_{\xi_{re}^1} \xrightarrow{\pi_1} B_{\xi^2} \xrightarrow{\varepsilon} B_{\xi_{re}^2} \dots$$

Clearly,  $P_w^1, P_w^2$  contain no empty states and  $\xi^{i+1} \in \widehat{\text{next}}(B_{\xi_{re}^i})$ ,  $\psi^{i+1} \in \widehat{\text{next}}(B_{\psi_{re}^i})$  for every  $i \geq 0$ . Taking into account Remark 6, we distinguish the following cases.

- (a) The set  $cl(\psi) \cap cl(\xi)$  contains no subformulas of the form  $\varphi_1 U \varphi_2$ .
- (b) For every  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$ ,  $\varphi_1 U \varphi_2$  does not appear in the scope of an always operator  $\square$  in at least one of  $\psi, \xi$ .
- (c) For every  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$  that is in the scope of an always operator  $\square$  in both  $\psi, \xi$ , there is an  $n \geq 0$ , such that the acceptance condition from  $\varphi_1 U \varphi_2$  is satisfied for every position  $n' \geq n$  in at least one of  $P_w^1, P_w^2$ .
- (d) There is at least one  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$  that is in the scope of an always operator  $\square$  in both  $\psi, \xi$ , and in both  $P_w^1, P_w^2$  the acceptance condition from  $\varphi_1 U \varphi_2$  is satisfied for infinitely many positions, and not satisfied for infinitely many positions, too.

If case (a), or (b), or (c) holds we act as follows. Inductively, we can determine a path  $P_w$  of next and  $\varepsilon$ -transitions of  $\mathcal{A}_\varphi$  over  $w$

$$P_w : B_{\varphi^0} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$

in the following way<sup>3</sup>: For  $\varphi = \varphi^0 = (\psi^0 \wedge \xi^0)_{re}$ , and for the next transitions  $(B_{\psi^0}, \pi_0, B_{\psi^1})$ ,  $(B_{\xi^0}, \pi_0, B_{\xi^1})$  we apply Lemma 12, and we obtain  $B_{\varphi^0} \neq \emptyset$ ,

<sup>3</sup>We shall call this inductive procedure, Procedure 2.

$\varphi^1 \in \text{next}(B_{\varphi^0})$  such that for every  $B_{\varphi^1}$  the triple  $(B_{\varphi^0}, \pi_0, B_{\varphi^1})$  is a next transition with

$$wt_1(B_{\psi_{re}^0}, \pi_0, B_{\psi^1}) \cdot wt_2(B_{\xi_{re}^0}, \pi_0, B_{\xi^1}) \leq wt(B_{\varphi^0}, \pi_0, B_{\varphi^1}).$$

We also get  $\bar{\xi}^{(1,1)} \in RULTL(K, AP)$  with  $\varphi_{re}^1 = \left(\psi_{re}^1 \wedge \bar{\xi}_{re}^{(1,1)}\right)_{re}$ , and an infinite sequence of next and  $\varepsilon$ -transitions  $B_{\xi_{re}^{(1,1)}} \xrightarrow{\pi_1} B_{\xi_{re}^{(1,2)}} \xrightarrow{*} B_{\xi_{re}^{(1,2)}} \dots$  with  $v_{B_{\xi_{re}^{(1,i)}}}(\bar{\xi}^{(1,i+1)}) = wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi_{re}^{i+1}})$  for every  $i \geq 1$ . Assume now that  $B_{\varphi^{j-1}}$  are built with the previous procedure for every  $j \leq m$ , which implies that there exists  $\bar{\xi}^{(m,1)} \in RULTL(K, AP)$  such that  $\varphi_{re}^m = \left(\psi_{re}^m \wedge \bar{\xi}_{re}^{(m,1)}\right)_{re}$ , and an infinite sequence of next and  $\varepsilon$ -transitions  $B_{\xi_{re}^{(m,1)}} \xrightarrow{\pi_m} B_{\xi_{re}^{(m,2)}} \xrightarrow{*} B_{\xi_{re}^{(m,2)}} \dots$  with  $v_{B_{\xi_{re}^{(m,i)}}}(\bar{\xi}^{(m,i+1)}) = wt_2(B_{\xi_{re}^{m-1+i}}, \pi_{m-1+i}, B_{\xi_{re}^{m+i}})$  for all  $i \geq 1$ . We apply Lemma 12 for  $\varphi_{re}^m = \left(\psi_{re}^m \wedge \bar{\xi}_{re}^{(m,1)}\right)_{re}$  and the next transitions  $(B_{\psi_{re}^m}, \pi_m, B_{\psi^{m+1}}), (B_{\xi_{re}^{(m,1)}}, \pi_m, B_{\xi_{re}^{(m,2)}})$ . We get  $B_{\varphi_{re}^m} \neq \emptyset, \varphi^{m+1} \in \text{next}(B_{\varphi_{re}^m})$  such that for every  $B_{\varphi^{m+1}}, (B_{\varphi_{re}^m}, \pi_m, B_{\varphi^{m+1}})$  is a next transition with

$$\begin{aligned} & wt_1(B_{\psi_{re}^m}, \pi_m, B_{\psi^{m+1}}) \cdot v_{B_{\xi_{re}^{(m,1)}}}(\bar{\xi}^{(m,2)}) \\ &= wt_1(B_{\psi_{re}^m}, \pi_m, B_{\psi^{m+1}}) \cdot wt_2(B_{\xi_{re}^m}, \pi_m, B_{\xi_{re}^{m+1}}) \\ &\leq wt(B_{\varphi_{re}^m}, \pi_m, B_{\varphi^{m+1}}). \end{aligned}$$

Hence, for every  $i \geq 0$  it holds  $wt_1(B_{\psi_{re}^i}, \pi_i, B_{\psi^{i+1}}) \cdot wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi_{re}^{i+1}}) \leq wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}})$ , which implies

$$\begin{aligned} & Val^\omega(wt_1(B_{\psi_{re}^i}, \pi_i, B_{\psi^{i+1}}) \cdot wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi_{re}^{i+1}}))_{i \geq 0} \\ &\leq Val^\omega(wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}))_{i \geq 0} \\ &\implies \\ & Val^\omega(wt_2(B_{\xi_{re}^i}, \pi_i, B_{\xi_{re}^{i+1}}))_{i \geq 0} \\ &\leq Val^\omega(wt(B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}))_{i \geq 0} \\ &\implies \end{aligned}$$

$$\begin{aligned} & Val^\omega (wt_1 (B_{\psi_{re}^i}, \pi_i, B_{\psi^{i+1}}))_{i \geq 0} \cdot Val^\omega (wt_2 (B_{\xi_{re}^i}, \pi_i, B_{\xi^{i+1}}))_{i \geq 0} \\ & \leq Val^\omega (wt (B_{\varphi_{re}^i}, \pi_i, B_{\varphi^{i+1}}))_{i \geq 0} \end{aligned}$$

where the first inequality holds by Lemma 3, and the third and fourth inequality are derived by the fact that  $wt_1 (B_{\psi_{re}^i}, \pi_i, B_{\psi^{i+1}}) = \mathbf{1}$  for every  $i \geq 0$ . Thus,

$$weight_{\mathcal{A}_\psi} (P_w^1) \cdot weight_{\mathcal{A}_\xi} (P_w^2) \leq weight_{\mathcal{A}_\varphi} (P_w).$$

Following the constructive proof of Lemma 12, and since either (a), or (b), or (c) holds, we get that for all  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$ ,  $P_w$  satisfies the acceptance condition for  $\varphi_1 U \varphi_2$  for infinitely many  $i \geq 0$ , i.e.,  $P_w$  is successful.

Assume now that case (d) holds, and let  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$  with the property of case (d). Clearly,  $\varphi_1 U \varphi_2$  is boolean. Let  $i_1 < i_2 < \dots$  be the sequence of positions with  $B_{\psi^{i_p}} \in F_{\varphi_1 U \varphi_2}$ ,  $p \geq 1$ , and with the additional property that in positions  $i_1 - 1 < i_2 - 1 < \dots$  the acceptance condition from  $\varphi_1 U \varphi_2$  is not satisfied. Then, due to the fact that  $\varphi_1 U \varphi_2$  is not in the scope of a next operator, we can determine a path  $\widehat{P}_w^2$  of next and  $\varepsilon$ -reduction transitions of  $\mathcal{A}_\xi$  over  $w$  such that for every position  $i_p$ ,  $p \geq 1$ , the acceptance condition of  $\varphi_1 U \varphi_2$  is satisfied, and  $weight_{\mathcal{A}_\psi} (P_w^1) \cdot weight_{\mathcal{A}_\xi} (P_w^2) \leq weight_{\mathcal{A}_\psi} (P_w^1) \cdot weight_{\mathcal{A}_\xi} (\widehat{P}_w^2)$ . Also,  $\widehat{P}_w^2$  can be chosen in such a way that the above statement is satisfied for every  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$  with the property of case (d).<sup>4</sup> We construct the path  $P_w \in next_{\mathcal{A}_\varphi} (w)$  by  $P_w^1$  and  $\widehat{P}_w^2$ , in the same way that  $P_w$  was constructed by  $P_w^1$  and  $P_w^2$  in cases (a), (b), (c). Then,  $P_w$  is successful and we get  $weight_{\mathcal{A}_\psi} (P_w^1) \cdot weight_{\mathcal{A}_\xi} (P_w^2) \leq weight_{\mathcal{A}_\psi} (P_w^1) \cdot weight_{\mathcal{A}_\xi} (\widehat{P}_w^2) \leq weight_{\mathcal{A}_\varphi} (P_w)$ .

We have shown that for every  $P_w^1 \in next_{\mathcal{A}_\psi} (w)$  with  $weight_{\mathcal{A}_\psi} (w) = \mathbf{1}$  and every  $P_w^2 \in next_{\mathcal{A}_\xi} (w)$  with  $weight_{\mathcal{A}_\xi} (w) \neq \mathbf{0}$  there is a  $P_w \in next_{\mathcal{A}_\varphi} (w)$  with

$$weight_{\mathcal{A}_\psi} (P_w^1) \cdot weight_{\mathcal{A}_\xi} (P_w^2) = weight_{\mathcal{A}_\xi} (P_w^2) \leq weight_{\mathcal{A}_\varphi} (P_w).$$

This implies that for every  $k_1 \in pri_{\mathcal{A}_\psi} (w) \setminus \{\mathbf{0}\}$ ,  $k_2 \in pri_{\mathcal{A}_\xi} (w) \setminus \{\mathbf{0}\}$  there

<sup>4</sup>The existence of this path is determined by Procedure 3, which is presented in the Appendix of this paper.

exists  $k \in \text{pri}_{\mathcal{A}_\varphi}(w)$  such that  $k_1 \cdot k_2 = k_2 \leq k$ , i.e.,

$$\begin{aligned}
 (\|\varphi\|, w) &= (\|\psi\|, w) \cdot (\|\xi\|, w) \\
 &= \left( \sum_{k_1 \in \text{pri}_{\mathcal{A}_\psi}(w) \setminus \{\mathbf{0}\}} k_1 \right) \cdot \left( \sum_{k_2 \in \text{pri}_{\mathcal{A}_\xi}(w) \setminus \{\mathbf{0}\}} k_2 \right) \\
 &= \left( \sum_{k_2 \in \text{pri}_{\mathcal{A}_\xi}(w) \setminus \{\mathbf{0}\}} k_2 \right) \\
 &\leq \left( \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w) \setminus \{\mathbf{0}\}} k \right) \\
 &= (\|\mathcal{A}_\varphi\|, w)
 \end{aligned}$$

as wanted. Hence, we have shown that  $(\|\varphi\|, w) \leq (\|\mathcal{A}_\varphi\|, w)$ , and  $(\|\mathcal{A}_\varphi\|, w) \leq (\|\varphi\|, w)$  for every  $w \in (\mathcal{P}(AP))^\omega$ , which implies that  $(\|\varphi\|, w) = (\|\mathcal{A}_\varphi\|, w)$ , and the proof is completed.  $\square$

The proof of the Lemma 14 can be found in the Appendix. Then, Lemma 15 can be proved with the same arguments with ones we used in the proof of Lemma 14.

**Lemma 14** *Let  $\psi U \xi \in RULTL(K, AP)$  with  $\psi, \xi \in stLTL(K, AP)$ , and  $\xi_j \in bLTL(K, AP)$  be reduced formulas ( $1 \leq j \leq k, k \geq 1$ ) and  $\pi \in \mathcal{P}(AP)$ . Let  $(B_\psi, \pi, B_{\psi'})$ ,  $(B_{\xi_j}, \pi, B_{\xi'_j})$  be next transitions with  $B_\psi \neq \emptyset, \psi' \in \widehat{\text{next}}(B_\psi)$ ,  $B_{\xi_j} \neq \emptyset, \xi'_j \in \widehat{\text{next}}(B_{\xi_j})$  ( $1 \leq j \leq k$ ), and  $v_{B_\psi}(\psi') \cdot \prod_{1 \leq j \leq k} v_{B_{\xi_j}}(\xi'_j) \neq \mathbf{0}$ .<sup>5</sup>*

*Then, for  $\varphi = \left( (\psi U \xi) \wedge \left( \bigwedge_{1 \leq j \leq k} \xi_j \right) \right)_{re}$  there exist  $B_\varphi \neq \emptyset, \varphi' \in \widehat{\text{next}}(B_\varphi)$ ,  $\psi'', \xi''_j \in bLTL(K, AP)$  ( $1 \leq j \leq k$ ) such that*

$$(i) \ (B_\varphi, \pi, B_{\varphi'}) \text{ is a next transition and } v_{B_\varphi}(\varphi') \geq v_{B_\psi}(\psi') \cdot \prod_{1 \leq j \leq k} v_{B_{\xi_j}}(\xi'_j),$$

---

<sup>5</sup>Since  $\xi_j$  ( $1 \leq j \leq k$ ) are boolean, by Remark 1, and the fact that  $k \cdot \mathbf{1} = \mathbf{1} \cdot k = k$ ,  $k \cdot \mathbf{0} = \mathbf{0} \cdot k = \mathbf{0}$  for every  $k \in K$ , we conclude that the product  $v_{B_\psi}(\psi') \cdot \prod_{1 \leq j \leq k} v_{B_{\xi_j}}(\xi'_j)$  is well defined.

(ii)  $\varphi'_{re} = \left( (\psi U \xi) \wedge \psi''_{re} \wedge \left( \bigwedge_{1 \leq j \leq k} (\xi''_j)_{re} \right) \right)_{re}$  and for every infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\psi^0} \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{\varepsilon} B_{\psi^1_{re}} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{\varepsilon} B_{\psi^2_{re}} \dots$$

with  $\psi^0 = \psi'_{re}$  (resp.  $\psi^0 = (\xi'_j)_{re}$ ,  $1 \leq j \leq k$ ) and  $v_{B_{\psi^i_{re}}}(\psi^{i+1}) \neq \mathbf{0}$  ( $i \geq 0$ ), there exists an infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\lambda^0} \xrightarrow{\pi_0} B_{\lambda^1} \xrightarrow{\varepsilon} B_{\lambda^1_{re}} \xrightarrow{\pi_1} B_{\lambda^2} \xrightarrow{\varepsilon} B_{\lambda^2_{re}} \dots$$

with  $\lambda^0 = \psi''_{re}$  (resp.  $\lambda^0 = (\xi''_j)_{re}$ ,  $1 \leq j \leq k$ ) and  $v_{B_{\lambda^i_{re}}}(\lambda^{i+1}) = v_{B_{\psi^i_{re}}}(\psi^{i+1})$  for every  $i \geq 0$ .

**Lemma 15** Let  $\psi U \xi \in RULTL(K, AP)$  with  $\psi, \xi \in stLTL(K, AP)$ , and  $\xi_j \in bLTL(K, AP)$  be reduced formulas ( $1 \leq j \leq k, k \geq 1$ ) and  $\pi \in \mathcal{P}(AP)$ . Let  $(B_\xi, \pi, B_{\xi'})$ ,  $(B_{\xi_j}, \pi, B_{\xi'_j})$  be next transitions with  $B_\xi \neq \emptyset$ ,  $\xi' \in \widehat{next}(B_\xi)$ ,  $B_{\xi_j} \neq \emptyset$ ,  $\xi'_j \in \widehat{next}(B_{\xi_j})$  ( $1 \leq j \leq k$ ), and  $v_{B_\xi}(\xi') \cdot \prod_{1 \leq j \leq k} v_{B_{\xi_j}}(\xi'_j) \neq \mathbf{0}$ .

Then, for  $\varphi = \left( (\psi U \xi) \wedge \left( \bigwedge_{1 \leq j \leq k} \xi_j \right) \right)_{re}$  there exist  $B_\varphi \neq \emptyset$ ,  $\varphi' \in \widehat{next}(B_\varphi)$ ,  $\xi'', \xi''_j \in bLTL(K, AP)$  ( $1 \leq j \leq k$ ) such that

(i)  $(B_\varphi, \pi, B_{\varphi'})$  is a next transition and  $v_{B_\varphi}(\varphi') \geq v_{B_\xi}(\xi') \cdot \prod_{1 \leq j \leq k} v_{B_{\xi_j}}(\xi'_j)$ ,

(ii)  $\varphi'_{re} = \left( \xi''_{re} \wedge \left( \bigwedge_{1 \leq j \leq k} (\xi''_j)_{re} \right) \right)_{re}$  and for every infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\psi^0} \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{\varepsilon} B_{\psi^1_{re}} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{\varepsilon} B_{\psi^2_{re}} \dots$$

with  $\psi^0 = \xi'_{re}$  (resp.  $\psi^0 = (\xi'_j)_{re}$ ,  $1 \leq j \leq k$ ) and  $v_{B_{\psi^i_{re}}}(\psi^{i+1}) \neq \mathbf{0}$  ( $i \geq 0$ ), there exists an infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\lambda^0} \xrightarrow{\pi_0} B_{\lambda^1} \xrightarrow{\varepsilon} B_{\lambda^1_{re}} \xrightarrow{\pi_1} B_{\lambda^2} \xrightarrow{\varepsilon} B_{\lambda^2_{re}} \dots$$

with  $\lambda^0 = \xi''_{re}$  (resp.  $\lambda^0 = (\xi''_j)_{re}$ ,  $1 \leq j \leq k$ ) and  $v_{B_{\lambda_{re}^i}}(\lambda^{i+1}) = v_{B_{\psi_{re}^i}}(\psi^{i+1})$  for every  $i \geq 0$ .

**Lemma 16** *Let  $\varphi = \psi U \xi$  with  $\psi, \xi \in stLTL(K, AP)$ . If  $\mathcal{A}_\psi, \mathcal{A}_\xi$  recognize  $\|\psi\|, \|\xi\|$  respectively, then  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** Let  $w = \pi_0 \pi_1 \pi_2 \dots \in (\mathcal{P}(AP))^\omega$  and  $\mathcal{A}_\psi = (Q_1, wt_1, I_1, \mathcal{F}_1)$ ,  $\mathcal{A}_\xi = (Q_2, wt_2, I_2, \mathcal{F}_2)$ , and  $\mathcal{A}_\varphi = (Q, wt, I, \mathcal{F})$ . Let also

$$P_w : B_{\varphi^0} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi_{re}^2} \dots$$

be a path in  $next_{\mathcal{A}_\varphi}(w)$  with  $weight_{\mathcal{A}_\varphi}(w) \neq \mathbf{0}$ . Since  $P_w$  is successful there is an  $l > 0$  such that  $B_{\varphi^l} \in F_\varphi$  for the first time. We claim that there are paths  $P_{w_{\geq j}}^1$  of  $\mathcal{A}_\psi$  over  $w_{\geq j}$ ,  $0 \leq j \leq l-1$ , and a path  $P_{w_{\geq l-1}}^1$  of  $\mathcal{A}_\xi$  over  $w_{\geq l-1}$ , that are simultaneously simulated while  $\mathcal{A}_\varphi$  runs  $P_w$ . This is due to the following. Until the  $l$ th next transition the automaton moves between states that are consistent sets of conjunctions containing  $\varphi$ . After the  $l$ th next transition the automaton moves between states that are consistent sets of conjunctions not containing  $\varphi$ . For every  $0 < j \leq l-1$ , at the  $j$ th next transition of  $P_w$  the choice of the next formula of the maximal  $\varphi$ -consistent subset of the state indicates a next transition of  $\mathcal{A}_\psi$  that can be considered as the first transition of a path of  $\mathcal{A}_\psi$  over the suffix of  $w$  starting at this point. At the  $l$ th next transition this choice indicates a next transition of  $\mathcal{A}_\xi$  that can be considered as the first of a path of  $\mathcal{A}_\xi$  over  $w_{\geq l-1}$ .

Now, formally for  $P_w$  we have that  $\varphi^0 = \varphi$ , and the following hold.

- For every  $0 < m < l$  there exist boolean formulas  $\varphi^{(m,1)}, \dots, \varphi^{(m,m)}$  such that  $\varphi_{re}^m = \left( \varphi \wedge \varphi_{re}^{(m,1)} \wedge \dots \wedge \varphi_{re}^{(m,m)} \right)_{re}$  with

(i)  $\varphi \wedge \varphi^{(m,1)} \in next \left( M_{B_{\varphi_{re}^{m-1}, \varphi}} \right)$ , and  $\varphi^{(m,1)} \in next \left( M_{B_{\varphi_{re}^{m-1}, \psi}} \right)$ , and

(ii)  $\varphi^{(m,p)} \in next \left( A^{(m-1,p-1)} \right)$  for some  $\varphi_{re}^{(m-1,p-1)}$ -consistent set  $A^{(m-1,p-1)}$  ( $2 \leq p \leq m$ ). Moreover,

$$\begin{aligned} & wt \left( B_{\varphi_{re}^{m-1}, \pi_{m-1}, B_{\varphi^m}} \right) \\ &= v_{M_{B_{\varphi_{re}^{m-1}, \psi}}} \left( \varphi^{(m,1)} \right) \cdot v_{A^{(m-1,1)}} \left( \varphi^{(m,2)} \right) \cdot \dots \cdot v_{A^{(m-1,m-1)}} \left( \varphi^{(m,m)} \right) \\ &= v_{M_{B_{\varphi_{re}^{m-1}, \psi}}} \left( \varphi^{(m,1)} \right) \end{aligned}$$

where last equality holds since  $weight_{\mathcal{A}_\varphi}(P_w) \neq \mathbf{0}$ , which implies that  $wt\left(B_{\varphi_{re}^{m-1}}, \pi_{m-1}, B_{\varphi^m}\right) \neq \mathbf{0}$ , i.e.,  $v_{A^{(m-1,j)}}(\varphi^{(m,j+1)}) = \mathbf{1}$  for every  $1 \leq j \leq m-1$ .<sup>6</sup>

- $\varphi_{re}^l = \left(\varphi_{re}^{(l,1)} \wedge \dots \wedge \varphi_{re}^{(l,l)}\right)_{re}$  for boolean formulas  $\varphi^{(l,1)}, \dots, \varphi^{(l,l)}$  with

(i)  $\varphi^{(l,1)} \in next\left(M_{B_{\varphi_{re}^{l-1}, \xi}}\right)$ , and

- (ii)  $\varphi^{(l,p)} \in next\left(A^{(l-1,p-1)}\right)$  for some  $\varphi_{re}^{(l-1,p-1)}$ -consistent set  $A^{(l-1,p-1)}$  ( $2 \leq p \leq l$ ). Furthermore,

$$\begin{aligned} & wt\left(B_{\varphi_{re}^{l-1}}, \pi_{l-1}, B_{\varphi^l}\right) \\ &= v_{M_{B_{\varphi_{re}^{l-1}, \psi}}}\left(\varphi^{(l,1)}\right) \cdot v_{A^{(l-1,1)}}\left(\varphi^{(l,2)}\right) \cdot \dots \cdot v_{A^{(l-1,l-1)}}\left(\varphi^{(l,l)}\right) \\ &= v_{M_{B_{\varphi_{re}^{l-1}, \psi}}}\left(\varphi^{(l,1)}\right). \end{aligned}$$

- Last, by induction on  $n$  and the same arguments used in the proof of Lemma 11 we get that for every  $n > l$  there exist boolean  $\varphi^{(n,1)}, \dots, \varphi^{(n,l)}$  such that  $\varphi_{re}^n = \left(\varphi_{re}^{(n,1)} \wedge \dots \wedge \varphi_{re}^{(n,l)}\right)_{re}$  and  $\varphi^{(n,p)} \in next\left(A^{(n-1,p)}\right)$  for some  $\varphi_{re}^{(n-1,p)}$ -consistent set  $A^{(n-1,p)}$  ( $1 \leq p \leq l$ ), and

$$\begin{aligned} & wt\left(B_{\varphi_{re}^{n-1}}, \pi_{n-1}, B_{\varphi^n}\right) \\ &= v_{A^{(n-1,1)}}\left(\varphi^{(n,1)}\right) \cdot v_{A^{(n-1,1)}}\left(\varphi^{(n,2)}\right) \cdot \dots \cdot v_{A^{(n-1,l-1)}}\left(\varphi^{(n,l)}\right) \\ &= \mathbf{1}, \end{aligned}$$

where the last equality is concluded by the fact that  $weight_{\mathcal{A}_\varphi}(P_w) \neq \mathbf{0}$ , i.e.,  $v_{A^{(n-1,j)}}(\varphi^{(n,j)}) = \mathbf{1}$  for every  $1 \leq j \leq l$ .

For every  $0 < m < l$  the sequence

$$\begin{aligned} P_{w_{\geq m-1}}^1 : M_{B_{\varphi_{re}^{m-1}, \psi}} &\xrightarrow{\pi_{m-1}} B_{\varphi^{(m,1)}} \xrightarrow{\varepsilon} A^{(m,1)} \xrightarrow{\pi_m} B_{\varphi^{(m+1,2)}} \xrightarrow{\varepsilon} A^{(m+1,2)} \dots \\ & A^{(l-1,l-m)} \xrightarrow{\pi_{l-1}} B_{\varphi^{(l,l-m+1)}} \xrightarrow{\varepsilon} A^{(l,l-m+1)} \xrightarrow{\pi_l} B_{\varphi^{(l+1,l-m+1)}} \dots \end{aligned}$$

<sup>6</sup>Recall that  $\varphi^{(m,1)}$  is boolean, as it is a next formula of an *LTL*-step formula. Then,  $\varphi^{(m,j)}$  ( $1 \leq j \leq m-1$ ) are boolean, since they are next formulas of boolean formulas.



is a path in  $next_{\mathcal{A}_\psi}(w_{\geq m-1})$  with

$$weight_{\mathcal{A}_\psi}(P_{w_{\geq m-1}}^1) = Val^\omega \left( v_{B_{\varphi_{re}^{m-1}, \psi}}(\varphi^{(m,1)}), \mathbf{1}, \mathbf{1}, \dots \right)$$

and the sequence

$$P_{w_{\geq l-1}}^2 : M_{B_{\varphi_{re}^{l-1}, \xi}} \xrightarrow{\pi_{l-1}} B_{\varphi^{(l,1)}} \xrightarrow{\varepsilon} A^{(l,1)} \xrightarrow{\pi_l} B_{\varphi^{(l+1,1)}} \xrightarrow{\varepsilon} A^{(l+1,1)} \dots$$

is a path of  $next_{\mathcal{A}_\xi}(w_{\geq l-1})$  with

$$weight_{\mathcal{A}_\xi}(P_{w_{\geq l-1}}^2) = Val^\omega \left( v_{B_{\varphi_{re}^{l-1}, \xi}}(\varphi^{(l,1)}), \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots \right).$$

We note that for every  $0 \leq j \leq l-1$ , and every  $i \geq 1$  the state  $B_{\varphi^{(j+i,i)}}$  appearing in the above paths could be any non-empty  $\varphi^{(j+i,i)}$ -consistent set. We show that  $P_{w_{\geq m}}^1$ ,  $0 \leq m \leq l-1$ , and  $P_{w_{\geq l-1}}^2$  are successful. Let us assume the contrary. Then, there exists a subformula of  $\varphi$  of the form  $\xi'U\xi''$  and an  $n \geq l$ , such that for every  $r > n$ , there is an  $1 \leq h \leq r$  such that  $\varphi^{(r,h)}$  does not satisfy the acceptance condition of  $\mathcal{A}_\xi$  corresponding to  $\xi'U\xi''$ , or it does not satisfy the acceptance condition of  $\mathcal{A}_\psi$  corresponding to  $\xi'U\xi''$ . But then  $P_w$  would not be successful, which is a contradiction. It holds

$$\begin{aligned} & weight_{\mathcal{A}_\varphi}(P_w) \\ &= Val^\omega \left( \begin{array}{c} v_{M_{B_{\varphi_{re}^0, \psi}}}(\varphi^{(0,1)}), \dots, v_{M_{B_{\varphi_{re}^{l-2}, \psi}}}(\varphi^{(l-1,1)}), \\ v_{M_{B_{\varphi_{re}^{l-1}, \xi}}}(\varphi^{(l,1)}), \mathbf{1}, \mathbf{1}, \dots \end{array} \right) \\ &= Val^\omega \left( \begin{array}{c} weight_{\mathcal{A}_\psi}(P_{w_{\geq 0}}^1), \dots, weight_{\mathcal{A}_\psi}(P_{w_{\geq l-2}}^1), \\ weight_{\mathcal{A}_\xi}(P_{w_{\geq l-1}}^2), \mathbf{1}, \mathbf{1}, \dots \end{array} \right) \\ &\leq \sum_{i \geq 0} \left( \begin{array}{c} \sum_{\substack{k_j \in pri_{\mathcal{A}_\psi}(w_{\geq j}) \\ k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i})}} Val^\omega(k_0, \dots, k_{i-1}, k_i, \mathbf{1}, \mathbf{1}, \dots) \end{array} \right) \\ &= \sum_{i \geq 0} \left( Val^\omega \left( \begin{array}{c} \sum_{k_0 \in pri_{\mathcal{A}_\psi}(w_{\geq 0})} k_0, \dots, \sum_{k_{i-1} \in pri_{\mathcal{A}_\psi}(w_{\geq i-1})} k_{i-1}, \\ \sum_{k_i \in pri_{\mathcal{A}_\xi}(w_{\geq i})} k_i, \mathbf{1}, \mathbf{1}, \dots \end{array} \right) \right) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i \geq 0} (Val^\omega (((\|\mathcal{A}_\psi\|, w_{\geq 0}), \dots, (\|\mathcal{A}_\psi\|, w_{\geq i-1}), (\|\mathcal{A}_\xi\|, w_{\geq i}), \mathbf{1}, \mathbf{1}, \dots))) \\
&= \sum_{i \geq 0} (Val^\omega (((\|\psi\|, w_{\geq 0}), \dots, (\|\psi\|, w_{\geq i-1}), (\|\xi\|, w_{\geq i}), \mathbf{1}, \mathbf{1}, \dots))) \\
&= (\|\varphi\|, w).
\end{aligned}$$

The second equality holds by Property 3, the inequality by Lemmas 1iii, 2, and the third equality by the distributivity of  $Val^\omega$  over finite sums, and the fact that  $pri_{\mathcal{A}_\psi}(w_{\geq j})$  ( $0 \leq j \leq i-1$ ),  $pri_{\mathcal{A}_\xi}(w_{\geq i})$  are finite for all  $i \geq 0$ . For every path  $P_w \in next_{\mathcal{A}_\varphi}(w)$  with  $weight_{\mathcal{A}_\varphi}(P_w) = \mathbf{0}$  it trivially holds  $weight_{\mathcal{A}_\varphi}(P_w) \leq (\|\varphi\|, w)$ . Thus, for every  $k \in pri_{\mathcal{A}_\varphi}(P_w)$ ,  $k \leq (\|\varphi\|, w)$ , and so by Lemmas 1ii, 2 we get

$$(\|\mathcal{A}_\varphi\|, w) = \sum_{k \in pri_{\mathcal{A}_\varphi}(w)} k \leq (\|\varphi\|, w).$$

We show now that  $(\|\varphi\|, w) \leq (\|\mathcal{A}_\varphi\|, w)$ . To this end, we fix an  $l \geq 0$ , and we let  $P_{w_{\geq m}}^1 \in next_{\mathcal{A}_\psi}(w_{\geq m})$  for every  $0 \leq m < l$ , and  $P_{w_{\geq l}}^2 \in next_{\mathcal{A}_\xi}(w_{\geq l})$ . We further assume that  $weight_{\mathcal{A}_\psi}(w_{\geq m}) \neq \mathbf{0}$  ( $0 \leq m < l$ ),  $weight_{\mathcal{A}_\xi}(w_{\geq l}) \neq \mathbf{0}$ . We prove that there exists a path  $P_w \in next_{\mathcal{A}_\varphi}(w)$  such that

$$\begin{aligned}
&Val^\omega \left( weight_{\mathcal{A}_\psi}(P_{w_{\geq 0}}^1), \dots, weight_{\mathcal{A}_\psi}(P_{w_{\geq l-1}}^1), weight_{\mathcal{A}_\xi}(P_{w_{\geq l}}^2), \mathbf{1}, \mathbf{1}, \dots \right) \\
&\leq weight_{\mathcal{A}_\varphi}(P_w).
\end{aligned}$$

We set

$$P_{w_{\geq m}}^1 : B_{\psi(m,0)} \xrightarrow{\pi_m} B_{\psi(m,1)} \xrightarrow{*} B_{\psi_{re}(m,1)} \xrightarrow{\pi_{m+1}} B_{\psi(m,2)} \dots$$

and

$$P_{w_{\geq l}}^2 : B_{\xi^0} \xrightarrow{\pi_l} B_{\xi^1} \xrightarrow{*} B_{\xi_{re}^1} \xrightarrow{\pi_{l+1}} B_{\xi^2} \dots$$

For every  $j \geq 1$ , it holds  $\psi^{(m,j)} \in \widehat{next}(B_{\psi_{re}^{(m,j-1)}})$  ( $0 \leq m \leq l-1$ ), and  $\xi^j \in \widehat{next}(B_{\xi_{re}^{j-1}})$ . We point out the following cases: (a) There is at least one subformula  $\varphi_1 U \varphi_2 \in cl(\psi)$  that is in the scope of an always operator in  $\psi$ , and for at least two of the paths  $P_{w_{\geq m}}^1$  ( $0 \leq m \leq l-1$ ) the acceptance condition from  $\varphi_1 U \varphi_2$  is satisfied for infinitely many positions, and not satisfied for infinitely many positions, too. (b) There is at least one  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$  that is in the scope of an always operator in both  $\psi, \xi$  and for at least two of the above  $l+1$  paths the acceptance condition

from  $\varphi_1 U \varphi_2$  is satisfied for infinitely many positions, and not satisfied for infinitely many positions, too.

First assume that cases (a) and (b) do not hold. Then, we set  $\varphi^0 = \varphi$  and the following is true.

- For every  $0 \leq m < l$ , with the use of Lemma 14, we obtain  $B_{\varphi_{re}^m} \neq \emptyset$ , and  $\varphi^{m+1} \in next(B_{\varphi_{re}^m})$  such that

$$\begin{aligned} B_{\varphi_{re}^m}(\varphi^{m+1}) &\geq \prod_{0 \leq j \leq m} v_{B_{\psi_{re}^{(j,m-j)}}}(\psi^{(j,m-j+1)}) \\ &= v_{B_{\psi_{re}^{(m,0)}}}(\psi^{(m,1)}) \end{aligned}$$

and  $(B_{\varphi_{re}^m}, \pi_m, B_{\varphi_{re}^{m+1}})$  is a next transition. More precisely,  $\varphi_{re}^m = \left( \varphi \wedge \left( \bigwedge_{0 \leq j \leq m-1} \psi^{(j,m-j)} \right) \right)_{re}$ , and we apply Lemma 14, for the next transitions  $(B_{\psi_{re}^{(m,0)}}, \pi_m, B_{\psi_{re}^{(m,1)}})$ , and  $(B_{\psi_{re}^{(j,m-j)}}, \pi_m, B_{\psi_{re}^{(j,m-j+1)}})$  ( $0 \leq j \leq m-1$ ).

- By Lemma 15 we obtain  $B_{\varphi_{re}^l} \neq \emptyset$  and  $\varphi^{l+1} \in next(B_{\varphi_{re}^l})$  such that  $(B_{\varphi_{re}^l}, \pi_l, B_{\varphi_{re}^{l+1}})$  is a next transition and

$$\begin{aligned} v_{B_{\varphi_{re}^l}}(\varphi^{l+1}) &\geq v_{B_{\xi^0}}(\xi^1) \cdot \prod_{0 \leq j \leq l-1} v_{B_{\psi_{re}^{(j,l-j)}}}(\psi^{(j,l-j+1)}) \\ &= v_{B_{\xi^0}}(\xi^1). \end{aligned}$$

It holds,  $\varphi_{re}^l = \left( \varphi \wedge \left( \bigwedge_{0 \leq i \leq l-1} \psi^{(i,l-i)} \right) \right)_{re}$ , and we apply Lemma 15, for the next transitions  $(B_{\psi_{re}^{(j,l-j)}}, \pi_m, B_{\psi_{re}^{(j,l-j+1)}})$  ( $0 \leq j \leq l-1$ ), and  $(B_{\xi^0}, \pi_l, B_{\xi^1})$ .

- Last, with the same arguments used in Lemma 12 we obtain, for every  $k > l$ ,  $B_{\varphi_{re}^k} \neq \emptyset$  and  $\varphi^{k+1} \in next(B_{\varphi_{re}^k})$  such that  $(B_{\varphi_{re}^k}, \pi_k, B_{\varphi_{re}^{k+1}})$

is a next transition and

$$\begin{aligned} v_{B_{\varphi_{re}^k}}(\varphi^{k+1}) &\geq v_{B_{\xi_{re}^{k-l}}}(\xi^{k-l+1}) \cdot \prod_{0 \leq j \leq l-1} v_{B_{\psi_{re}^{(j,k-j)}}}(\psi^{(j,k-j+1)}) \\ &= \mathbf{1}, \end{aligned}$$

where the last equality is obtained by the following. It holds  $\psi, \xi \in stLTL(K, AP)$ , which implies that  $\psi_{re}^{(m,j)}, \xi_{re}^j \in bLTL(K, AP)$  for all  $j \geq 1, 0 \leq m \leq l-1$ . Since  $P_{w \geq m}^1$  ( $0 \leq m \leq l-1$ ), and  $P_{w \geq l}^2$  have non-zero weight, all but the first next transitions appearing in  $P_{w \geq m}^1$  ( $0 \leq m \leq l-1$ ), and  $P_{w \geq l}^2$  have weight  $\mathbf{1}$ .

Clearly, the path  $P_w : B_{\varphi^0} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{\varepsilon} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \dots$  (where we let  $B_{\varphi^i}$  be any non-empty  $\varphi^i$ -consistent set ( $i \geq 0$ )) is a successful path of  $\mathcal{A}_\varphi$  over  $w$ . This is concluded by the constructive proofs of Lemmas 14, 15, 12 and by the fact that (a), and (b) do not hold, which imply that for all  $\varphi_1 U \varphi_2 \in cl(\psi) \cap cl(\xi)$ ,  $P_w$  satisfies the acceptance condition for  $\varphi_1 U \varphi_2$  for infinitely many  $i \geq 0$ . It holds

$$wt_1 \left( B_{\psi_{re}^{(m,0)}}, \pi_m, B_{\psi^{(m,1)}} \right) \leq wt \left( B_{\varphi_{re}^m}, \pi_m, B_{\varphi^{m+1}} \right)$$

for every  $0 \leq m < l$ , and

$$wt_2 \left( B_{\xi_{re}^0}, \pi_l, B_{\xi^1} \right) \leq wt \left( B_{\varphi_{re}^l}, \pi_l, B_{\varphi^{l+1}} \right)$$

and

$$\mathbf{1} \leq wt \left( B_{\varphi_{re}^k}, \pi_k, B_{\varphi^{k+1}} \right)$$

for every  $k > l$ . Hence, by the above relations, Lemma 3, and Property 3 we get

$$\begin{aligned} Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P_{w \geq 0}^1 \right), \dots, weight_{\mathcal{A}_\psi} \left( P_{w \geq l-1}^1 \right), weight_{\mathcal{A}_\xi} \left( P_{w \geq l}^2 \right), \mathbf{1}, \mathbf{1}, \dots \right) \\ \leq weight_{\mathcal{A}_\varphi} (P_w). \end{aligned}$$

Now, if case (a) or (b) holds, we can prove our claim following the same arguments used in the proof of Lemma 13. Thus, for every  $l \geq 0$ , every  $P_{w \geq m}^1 \in next_{\mathcal{A}_\psi}(w_{\geq m})$ , where  $0 \leq m < l$ , and every  $P_{w \geq l}^2 \in next_{\mathcal{A}_\xi}(w_{\geq l})$ , there exists a  $P_w \in next_{\mathcal{A}_\varphi}(w)$  such that

$$\begin{aligned} Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P_{w \geq 0}^1 \right), \dots, weight_{\mathcal{A}_\psi} \left( P_{w \geq l-1}^1 \right), weight_{\mathcal{A}_\xi} \left( P_{w \geq l}^2 \right), \mathbf{1}, \mathbf{1}, \dots \right) \\ \leq weight_{\mathcal{A}_\varphi} (P_w). \end{aligned}$$

Thus, it holds

$$\begin{aligned}
 & \sum_{\substack{l \geq 0, 0 \leq m < l \\ k_m \in \text{pri}_{\mathcal{A}_\psi}(w_{\geq m}) \\ k_l \in \text{pri}_{\mathcal{A}_\xi}(w_{\geq l})}} \text{Val}^\omega(k_0, \dots, k_{l-1}, k_l, \mathbf{1}, \mathbf{1}, \dots) \leq \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k \\
 & \implies \\
 & \sum_{l \geq 0} \text{Val}^\omega \left( \begin{array}{c} \sum_{k_0 \in \text{pri}_{\mathcal{A}_\psi}(w_{\geq 0})} k_0, \dots, \\ \sum_{k_{l-1} \in \text{pri}_{\mathcal{A}_\psi}(w_{\geq l-1})} k_{l-1}, \quad \sum_{k_l \in \text{pri}_{\mathcal{A}_\xi}(w_{\geq l})} k_l, \\ \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots \end{array} \right) \leq \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k \\
 & \implies \\
 & \sum_{l \geq 0} \text{Val}^\omega \left( \begin{array}{c} (\|\psi\|, w_{\geq 0}), \dots, (\|\psi\|, w_{\geq l-1}), (\|\varphi\|, w_{\geq l}), \\ \mathbf{1}, \mathbf{1}, \mathbf{1}, \dots \end{array} \right) \leq \sum_{k \in \text{pri}_{\mathcal{A}_\varphi}(w)} k \\
 & \implies \\
 & (\|\varphi\|, w) \leq (\|\mathcal{A}_\varphi\|, w),
 \end{aligned}$$

where the second inequality is obtained by the distributivity of  $\text{Val}^\omega$  over finite sums, and the fact that  $\text{pri}_{\mathcal{A}_\psi}(w_{\geq m})$  ( $0 \leq m < l$ ), and  $\text{pri}_{\mathcal{A}_\xi}(w_{\geq l})$  are finite, and this concludes our proof.  $\square$

**Lemma 17** *Let  $\psi \in \text{stLTL}(K, AP)$  such that  $\varphi = \square\psi$ . If  $\mathcal{A}_\psi$  recognizes  $\|\psi\|$ , then  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** Let  $\mathcal{A}_\psi = (Q', wt', I', \mathcal{F}')$ ,  $\mathcal{A}_\varphi = (Q, wt, I, \mathcal{F})$ . First, we prove that  $(\|\mathcal{A}_\varphi\|, w) \leq (\|\varphi\|, w)$  for every  $w \in (\mathcal{P}(AP))^\omega$ . To this end, let  $w = \pi_0\pi_1\pi_2\dots$  and  $P_w \in \text{next}_{\mathcal{A}_\varphi}(w)$  be a path with  $\text{weight}_{\mathcal{A}_\varphi}(P_w) \neq \mathbf{0}$ . We show that there exist paths  $P'_{w_{\geq i}} \in \text{next}_{\mathcal{A}_\psi}(w_{\geq i})$  ( $i \geq 0$ ) such that

$$\text{weight}_{\mathcal{A}_\varphi}(P_w) \leq \text{Val}^\omega \left( \text{weight}_{\mathcal{A}_\psi}(P'_{w_{\geq i}}) \right)_{i \geq 0}$$

Without any loss we may assume that  $P_w$  starts with a next transition. So we let

$$P_w : B_{\varphi^0} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{*} B_{\varphi^1_{re}} \xrightarrow{\pi_1} B_{\varphi^2} \xrightarrow{*} B_{\varphi^2_{re}} \rightarrow \dots$$

It holds  $\varphi^0 = \varphi_{re}^0 = \varphi$  and for every  $i \geq 1$ , we can prove by induction on  $i$  and the same arguments used in Lemma 12, that there exist boolean formulas  $\varphi^{(i,1)}, \dots, \varphi^{(i,i)}$  such that  $\varphi_{re}^i = \left( \varphi \wedge \varphi_{re}^{(i,1)} \wedge \dots \wedge \varphi_{re}^{(i,i)} \right)_{re}$ , and  $\varphi_{re}^{(i,1)} \in next \left( M_{B_{\varphi_{re}^{i-1}, \psi}} \right)$ ,  $\varphi^{(i,p)} \in next \left( A^{(i-1,p-1)} \right)$  where  $A^{(i-1,p-1)}$  is a  $\varphi_{re}^{(i-1,p-1)}$ -consistent set ( $2 \leq p \leq i$ ), and

$$\begin{aligned} wt \left( B_{\varphi_{re}^{i-1}, \pi_{i-1}, B_{\varphi^i}} \right) &= v_{M_{B_{\varphi_{re}^{i-1}, \psi}}} \left( \varphi^{(i,1)} \right) \cdot \prod_{2 \leq p \leq i} v_{A^{(i-1,p-1)}} \left( \varphi^{(i,p)} \right) \quad (6) \\ &= v_{M_{B_{\varphi_{re}^{i-1}, \psi}}} \left( \varphi^{(i,1)} \right) \end{aligned}$$

where the last equality holds since  $v_{A^{(i-1,1)}} \left( \varphi^{(i,2)} \right) = \dots = v_{A^{(i-1,i-1)}} \left( \varphi^{(i,i)} \right) = \mathbf{1}$ .

Hence, for every  $i \geq 0$  we can define the path  $P'_{w_{\geq i}} \in next_{\mathcal{A}_\psi} (w_{\geq i})$  as follows.

$$P'_{w_{\geq i}} : M_{B_{\varphi_{re}^i, \psi}} \xrightarrow{\pi_i} B_{\varphi^{(i+1,1)}} \xrightarrow{\varepsilon} A^{(i+1,1)} \xrightarrow{\pi_{i+1}} B_{\varphi^{(i+2,2)}} \xrightarrow{\varepsilon} A^{(i+2,2)} \dots$$

where for every  $j \geq 1$ , we let  $B_{\varphi^{(i+j,j)}}$  be any non-empty  $\varphi^{(i+j,j)}$ -consistent set. We show that  $P'_{w_{\geq i}}$  is successful. Let us assume the contrary. This means that there exists a boolean subformula of  $\psi$  of the form  $\xi U \xi'$  and an  $n \geq 0$ , such that for every  $r > n$ , there is an  $1 \leq h \leq r$  such that  $\varphi^{(r,h)}$  does not satisfy the acceptance condition of  $\mathcal{A}_\psi$  corresponding to  $\xi' U \xi''$ . But then  $P_w$  would not be successful, which is a contradiction. Moreover, it holds

$$\begin{aligned} &weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq i}} \right) \\ &= Val^\omega \left( v_{M_{B_{\varphi_{re}^i, \psi}}} \left( \varphi^{(i+1,1)} \right), v_{A^{(i+1,1)}} \left( \varphi^{(i+2,2)} \right), v_{A^{(i+2,2)}} \left( \varphi^{(i+3,3)} \right), \dots \right) \\ &= Val^\omega \left( v_{M_{B_{\varphi_{re}^i, \psi}}} \left( \varphi^{i+1,1} \right), \mathbf{1}, \mathbf{1}, \dots \right) \\ &= v_{M_{B_{\varphi_{re}^i, \psi}}} \left( \varphi^{i+1,1} \right), \end{aligned}$$

where the last equality holds by Property 3. Then,

$$\begin{aligned} weight_{\mathcal{A}_\varphi} (P_w) &= Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 0}} \right), weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 1}} \right), \dots \right) \\ &\leq Val^\omega \left( \sum_{k_0 \in pri_{\mathcal{A}_\psi}(w_{\geq 0})} k_0, \dots \right) \end{aligned}$$

$$\begin{aligned} &= Val^\omega ((\|\psi\|, w_{\geq 0}), (\|\psi\|, w_{\geq 1}), \dots) \\ &= (\|\varphi\|, w), \end{aligned}$$

where the first inequality is concluded by Lemmas 1iii, 3.

Hence, for every  $k \in pri_{\mathcal{A}_\varphi}(w)$  it holds  $k \leq (\|\varphi\|, w)$ , and thus using Lemmas 1ii, 3 we get  $(\|\mathcal{A}_\varphi\|, w) = \sum_{k \in pri_{\mathcal{A}_\varphi}(w)} k \leq (\|\varphi\|, w)$ . We show now that  $(\|\varphi\|, w) \leq (\|\mathcal{A}_\varphi\|, w)$ . To this end, we let  $P'_{w_{\geq i}} \in next_{\mathcal{A}_\psi}(w_{\geq i})$  ( $i \geq 0$ ). We will prove that there exists a  $P_w \in next_{\mathcal{A}_\varphi}(w)$  with

$$Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 0}} \right), weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 1}} \right), \dots \right) \leq weight_{\mathcal{A}_\varphi} (P_w).$$

If  $Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 0}} \right), weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 1}} \right), \dots \right) = \mathbf{0}$ , then the inequality holds for every  $P_w \in next_{\mathcal{A}_\varphi}(w)$ . Otherwise, no empty states appear in  $P'_{w_{\geq i}}$  ( $i \geq 0$ ) and the subsequent hold.

Let  $\varphi_1 U \varphi_2 \in cl(\psi)$ . There exist paths  $\widehat{P}_{w_{\geq i}} \in next_{\mathcal{A}_\psi}(w_{\geq i})$  ( $i \geq 1$ )<sup>7</sup> with the following properties. (a) There are infinitely many  $j \geq 1$  such that at the next transition of  $\widehat{P}_{w_{\geq k}}$  ( $1 \leq k \leq j$ ) that processes the letter  $\pi_j$ , and at the corresponding next transition of  $P'_{w_{\geq 0}}$ , the automaton moves to a state that satisfies the acceptance condition of  $\varphi_1 U \varphi_2$ , and (b)

$$\begin{aligned} &Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 0}} \right), weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 1}} \right), \dots \right) \\ &\leq Val^\omega \left( weight_{\mathcal{A}_\psi} \left( P'_{w_{\geq 0}} \right), weight_{\mathcal{A}_\psi} \left( \widehat{P}_{w_{\geq 1}} \right), weight_{\mathcal{A}_\psi} \left( \widehat{P}_{w_{\geq 2}} \right), \dots \right) \end{aligned}$$

Moreover, the paths  $\widehat{P}_{w_{\geq i}}$  ( $i \geq 1$ ) can be chosen so that condition (a) is satisfied for every  $\varphi_1 U \varphi_2 \in cl(\psi)$ . We set  $\widehat{P}_{w_{\geq 0}} = P'_{w_{\geq 0}}$ , and for every  $i \geq 0$  we let

$$\widehat{P}_{w_{\geq i}} : B_{\psi^{(i,0)}} \xrightarrow{\pi_i} B_{\psi^{(i,1)}} \xrightarrow{*} B_{\psi_{re}^{(i,1)}} \xrightarrow{\pi_{i+1}} B_{\psi^{(i,2)}} \dots$$

Clearly, for every  $j \geq 1$  it holds  $\psi^{(i,j)} \in \widehat{next} \left( B_{\psi_{re}^{(i,j-1)}} \right)$  ( $i \geq 0$ ). Then, we set  $\varphi^0 = \varphi$  and with the same procedure used in Lemma 14 we obtain

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<sup>7</sup>We can prove for every  $i \geq 1$  the existence of the path  $\widehat{P}_{w_{\geq i}}$  following the constructive arguments of Procedure 3.

for every  $m \geq 0$ , a  $B_{\varphi_{re}^m} \neq \emptyset$  and  $\varphi^{m+1} \in next(B_{\varphi_{re}^m})$ , such that

$$\begin{aligned} v_{B_{\varphi_{re}^m}}(\varphi^{m+1}) &\geq \prod_{0 \leq j \leq m} v_{B_{\psi_{re}^{(j,m-j)}}}(\psi^{(j,m-j+1)}) \\ &= \prod_{0 \leq j \leq m} wt'(B_{\psi_{re}^{(j,m-j)}}, \pi_m, B_{\psi^{(j,m-j+1)}}) \\ &= wt'(B_{\psi_{re}^{(m,0)}}, \pi_m, B_{\psi^{(m,1)}}) \\ &= weight_{\mathcal{A}_\psi}(\widehat{P}_{w_{\geq m}}). \end{aligned}$$

Then, the path

$$P_w : B_{\varphi^0} \xrightarrow{\pi_0} B_{\varphi^1} \xrightarrow{\varepsilon} B_{\varphi_{re}^1} \xrightarrow{\pi_1} B_{\varphi^2} \dots$$

(where we let  $B_{\varphi^1}$  be any non-empty  $\varphi^1$ -consistent set ( $i \geq 0$ )) is a successful path of next and  $\varepsilon$ -reduction transitions of  $\mathcal{A}_\varphi$  over  $w$  and it holds

$$\begin{aligned} Val^\omega \left( weight_{\mathcal{A}_\psi}(\widehat{P}_{w_{\geq i}}) \right)_{i \geq 0} &\leq Val^\omega \left( v_{B_{\varphi_{re}^i}}(\varphi^{i+1}) \right)_{i \geq 0} \\ &= weight_{\mathcal{A}_\varphi}(P_w). \end{aligned}$$

Thus, for every family  $P'_{w_{\geq i}} \in next_{\mathcal{A}_\psi}(w_{\geq i})$  ( $i \geq 0$ ) there exists a  $P_w \in next_{\mathcal{A}_\varphi}(w)$  such that

$$Val^\omega \left( weight_{\mathcal{A}_\psi}(P'_{w_{\geq i}}) \right)_{i \geq 0} \leq weight_{\mathcal{A}_\varphi}(P_w)$$

i.e., for every family  $k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i})$  ( $i \geq 0$ ) there exists a  $k \in pri_{\mathcal{A}_\varphi}(w)$  such that  $Val^\omega(k_i)_{i \geq 0} \leq k$ . Then, by Lemma 2 we get

$$\sum_{\substack{k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i}) \\ i \geq 0}} Val^\omega(k_i)_{i \geq 0} \leq \sum_{k \in pri_{\mathcal{A}_\varphi}(w)} k = (\|\mathcal{A}_\varphi\|, w)$$

Moreover,

$$\begin{aligned} \sum_{\substack{k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i}) \\ i \geq 0}} Val^\omega(k_i)_{i \geq 0} &= Val^\omega \left( \sum_{k_i \in pri_{\mathcal{A}_\psi}(w_{\geq i})} k_i \right)_{i \geq 0} \\ &= Val^\omega(\|\psi\|, w_{\geq i})_{i \geq 0} \\ &= (\|\varphi\|, w) \end{aligned}$$



where the first equality holds by the distributivity of  $Val^\omega$  over finite sums and the fact that for every  $i \geq 0$  the set  $pri_{\mathcal{A}_\psi}(w_{\geq i})$  is finite. We conclude that  $(\|\varphi\|, w) \leq (\|\mathcal{A}_\varphi\|, w)$ . Hence, for every  $w \in (\mathcal{P}(AP))^\omega$  we have  $(\|\mathcal{A}_\varphi\|, w) = (\|\varphi\|, w)$ , and the proof is completed.  $\square$

**Lemma 18** *Let  $\varphi \in bLTL(K, AP)$ . Then,  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** We prove our claim by induction on the structure of  $bLTL(K, AP)$ -formulas and using the same arguments as in Lemmas 7, 8, 9, 13, 16, 17.  $\square$

**Lemma 19** *Let  $\varphi \in stLTL(K, AP)$ . Then,  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** Our claim is derived by Lemmas 18, 7, 8.  $\square$

**Theorem 1** *Let  $\varphi \in RULTL(K, AP)$ . Then,  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Proof:** By Lemmas 7, 9, 8, 13, 19, 16, 17, 18 we get that  $\|\mathcal{A}_\varphi\| = \|\varphi\|$  for every  $\varphi \in RULTL(K, AP)$ .  $\square$

**Corollary 1** *Let  $\varphi \in RULTL(K, AP)$ . Then, we can effectively construct a wBa over  $\mathcal{P}(AP)$  and  $K$  recognizing  $\|\varphi\|$ .*

## 7 Weighted LTL over Generalized Product $\omega$ -valuation Monoids

We let  $AP$  be a finite set of atomic propositions and  $K = (K, +, \cdot, Val^\omega, \mathbf{0}, \mathbf{1})$  be an idempotent generalized product  $\omega$ -valuation monoid. The syntax and semantics over the weighted  $LTL$  over  $AP$  and  $K$  is defined as in the previous section, as well as the class  $LTL(K, AP)$ , and the fragment of  $bLTL(K, AP)$ . We let a *restricted LTL-step formula* be an  $LTL(K, AP)$ -formula of the form  $\bigvee_{1 \leq i \leq n} (k_i \wedge \varphi_i)$  with  $k_i \in K \setminus \{\mathbf{0}, \mathbf{1}\}$  and  $\varphi_i \in bLTL(K, AP)$  for every  $1 \leq i \leq n$ . We denote by  $r-stLTL(K, AP)$  the class of restricted  $LTL$ -step formulas over  $AP$  and  $K$ . We introduce now the syntactic fragment of *totally restricted U-nesting LTL*-formulas.

**Definition 11** *The fragment of totally restricted U-nesting LTL-formulas over  $AP$  and  $K$ , denoted by  $t-RULTL(K, AP)$ , is the least class of formulas in  $LTL(K, AP)$  which is defined inductively in the following way.*

- $k \in t\text{-RULTL}(K, AP)$  for every  $k \in K$ .
- $bLTL(K, AP) \subseteq t\text{-RULTL}(K, AP)$
- If  $\varphi \in t\text{-RULTL}(K, AP)$ , then  $\bigcirc\varphi \in t\text{-RULTL}(K, AP)$ .
- If  $\varphi, \psi \in t\text{-RULTL}(K, AP)$ , then  $\varphi \vee \psi \in t\text{-RULTL}(K, AP)$ .
- If  $\varphi \in bLTL(K, AP)$  and  $\psi \in r\text{-stLTL}(K, AP)$ ,  
or  $\psi \in bLTL(K, AP)$ , or  $\psi = \xi U \lambda$ , or  $\psi = \Box \xi$   
where  $\xi, \lambda \in r\text{-stLTL}(K, AP)$ ,  
then  $\varphi \wedge \psi, \psi \wedge \varphi \in t\text{-RULTL}(K, AP)$ .
- If  $\varphi, \psi \in r\text{-stLTL}(K, AP)$ , then  $\varphi U \psi \in t\text{-RULTL}(K, AP)$ .
- If  $\varphi \in r\text{-stLTL}(K, AP)$ , then  $\Box \varphi \in t\text{-RULTL}(K, AP)$ .

We adopt the theory of the previous section (observe that using induction on the structure of  $\varphi$  we can derive that for every  $\varphi \in t\text{-RULTL}(K, AP)$  and  $\varphi$ -consistent set  $B_\varphi$ , it holds  $\text{next}(B_\varphi) \subseteq t\text{-RULTL}(K, AP)$ ). The following theorem is obtained by induction on the structure of  $t\text{-RULTL}(K, AP)$ -formulas and using the same arguments as the ones used in Lemmas 7, 9, 8, 13, 19, 16, 17, 18 of the previous section. The stronger syntactical restriction that we impose on the fragment of *totally restricted U-nesting LTL*-formulas allows us the use of the distributivity of  $Val^\omega$  over finite sums for generalized product  $\omega$ -valuation monoids and Lemma 3 whenever necessary.

**Theorem 2** *Let  $\varphi \in t\text{-RULTL}(K, AP)$ . Then,  $\mathcal{A}_\varphi$  recognizes  $\|\varphi\|$ .*

**Example 6** *Let  $AP = \{a, b\}$ , and  $\varphi = \Box(a \wedge 2) \in t\text{-RULTL}(K, AP)$  where  $K$  is the generalized product  $\omega$ -valuation monoid of Example 2. Then,  $\mathcal{A}_\varphi = (Q, wt, in, \mathcal{F})$  is defined below, where  $\pi$  ranges over  $\mathcal{P}(AP)$ , and by  $\pi_a$  we denote any letter in  $\mathcal{P}(AP)$  that contains  $a$ .*

- $Q = \{q_1, \dots, q_5\}$  with  $q_1 = \emptyset$ ,  $q_2 = \{-\infty\}$ ,  $q_3 = \{\text{true}\}$ ,  $q_4 = \{\varphi, a \wedge 2, a, 2\}$ ,  $q_5 = \{\varphi \wedge (\text{true} \wedge \text{true}), \varphi, a \wedge 2, 2, a, \text{true} \wedge \text{true}, \text{true}\}$
- The states with initial weight  $\infty$  are the sets  $q_1, q_4$ .
- The transitions with weight  $\neq -\infty$  are the following:  
 $wt(q_3, \pi, q_3) = wt(q_5, \varepsilon, q_4) = wt(q_k, \varepsilon, q_k) = \infty$  where  $k = 2, 3, 4$ , and  
 $wt(q_4, \pi_a, q_5) = 2$ .

- *The automaton has no final sets since  $\varphi$  contains no  $U$  operators.*

**Corollary 2** *Let  $\varphi \in t\text{-RULTL}(K, AP)$ . Then, we can effectively construct a  $wBa$  over  $\mathcal{P}(AP)$  and  $K$  recognizing  $\|\varphi\|$ .*

## 8 Conclusion

In this paper we introduced a weighted *LTL* over product  $\omega$ -valuation monoids (resp. generalized product  $\omega$ -valuation monoids) that satisfy specific properties, and proved that for every formula  $\varphi$  of a syntactic fragment of the weighted *LTL* we can effectively construct a weighted generalized Büchi automaton with  $\varepsilon$ -transitions  $\mathcal{A}_\varphi$  whose behavior coincides with the semantics of  $\varphi$ . We provided in this way a theoretical basis for the definition of quantitative model-checking algorithms. The structure of product  $\omega$ -valuation monoids and generalized product  $\omega$ -valuation monoids, that was used for the domain of weights, refers to an interesting range of possible applications. Naturally, in order to reach the goal of quantitative reasoning it is necessary to further investigate complexity and decidability results, providing in this way more arguments for the definition of model-checking algorithms incorporating the proposed weighted *LTL*. In [29] the authors introduced the notion of safety in the weighted setting. More precisely, for a rational number  $q$ , a finite series  $s$  over a given alphabet and  $\mathbb{Q}$  is called  $q$ -safe if every word with coefficient at least  $q$  has a prefix all whose extensions have coefficient at least  $q$ . Given a deterministic weighted automaton, the authors relate the safety of its behavior with its structure. They also propose the extension of their theory to infinite words as a challenging perspective and we further add that the definition of the notion of safety for infinitary series could be related with syntactical fragments of the weighted *LTL* and the structural properties of the corresponding weighted generalized Büchi automaton with  $\varepsilon$ -transitions that we propose in this paper. Finally, another interesting road for extending the theory of our weighted *LTL*, is to study its relation with weighted *FO* logic,  $\omega$ -star-free series and weighted counter-free automata on infinite words.

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## Appendix

In the following proof, we prove that the structure presented in Example 2 is indeed a generalized product  $\omega$ -valuation monoid.

**Proof:** We prove first the distributivity of  $Val^\omega$  over finite sums for generalized product  $\omega$ -valuation monoids.

Let  $L \subseteq_{fin} \overline{\mathbb{R}}$ , finite index sets  $I_j$  ( $j \geq 0$ ), and  $k_{i_j} \in L$  ( $i_j \in I_j, j \geq 0$ ) such that for all but a finite number of  $j \geq 0$ , it holds  $k_{i_j} \in L \setminus \{\infty, -\infty\}$  for all  $i_j \in I_j$ , or  $k_{i_j} \in \{\infty, -\infty\}$  for all  $i_j \in I_j$ . We will prove that

$$\liminf \left( \sup_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}} = \sup_{(i_j)_j \in I_0 \times I_1 \times \dots} \left( \liminf (k_{i_j})_{j \in \mathbb{N}} \right).$$

We set  $A = \liminf \left( \sup_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}}$ , and  $B = \sup_{(i_j)_j \in I_0 \times I_1 \times \dots} \left( \liminf (k_{i_j})_{j \in \mathbb{N}} \right)$ .

Assume that there exists an  $l \geq 0$ , such that  $k_{i_l} = -\infty$  for all  $i_l \in I_l$ . Then, for all  $(i_j)_j \in I_0 \times I_1 \times \dots$ ,  $\liminf (k_{i_j})_{j \in \mathbb{N}} = -\infty$ , i.e.,  $B = -\infty$ . Moreover,  $\sup_{i_l \in I_l} k_{i_l} = -\infty$ , and thus  $A = -\infty$  as wanted. Otherwise, we point out the following cases:

(I) Assume that for all  $j \geq 0$  there exists  $i_j \in I_j$  such that  $k_{i_j} = \infty$ . Then, there exist  $(i_j)_j \in I_0 \times I_1 \times \dots$  such that  $\liminf (k_{i_j})_{j \in \mathbb{N}} = \infty$  which implies that  $B = \infty$ . In addition, we get that  $\sup_{i_j \in I_j} k_{i_j} = \infty$  for all  $j \geq 0$ , i.e.,  $A = \infty$  as well.

(II) Assume that there exists finitely many  $j \geq 0$ , such that  $k_{i_j} \neq \infty$  for all  $i_j \in I_j$ , then  $\sup_{i_j \in I_j} k_{i_j} \neq \infty$  for only a finite number of  $j \geq 0$ , which

implies that  $A = \inf \left\{ \sup_{i_j \in I_j} k_{i_j} \mid j \geq 0 \text{ with } \sup_{i_j \in I_j} k_{i_j} \neq \infty \right\}$ . Moreover,

for all but a finite number of  $j \geq 0$  it holds  $k_{i_j} \in L \setminus \{\infty, -\infty\}$  for all  $i_j \in I_j$ , or  $k_{i_j} \in \{\infty, -\infty\}$  for all  $i_j \in I_j$ , thus there exists finitely many  $j \geq 0$  such that  $k_{i_j} \in L \setminus \{\infty, -\infty\}$  for all  $i_j \in I_j$ , which implies that the following equalities are true.

$$B = \sup_{\substack{(i_j)_j \in I_0 \times I_1 \times \dots \\ k_{i_j} \neq -\infty, j \geq 0}} \left( \liminf (k_{i_j})_{j \in \mathbb{N}} \right)$$



$$\begin{aligned}
 &= \sup_{\substack{(i_j)_{j \in I_0 \times I_1 \times \dots} \\ k_{i_j} \neq -\infty, j \geq 0}} (\inf \{k_{i_j} \mid j \geq 0 \text{ with } k_{i_j} \neq \infty\}) \\
 &= \inf \left\{ \sup_{i_j \in I_j} k_{i_j} \mid j \geq 0 \text{ with } \sup_{i_j \in I_j} k_{i_j} \neq \infty \right\} = A.
 \end{aligned}$$

(III) Assume that there exists infinitely many  $j \geq 0$  such that  $k_{i_j} \neq \infty$  for all  $i_j \in I_j$ . Since  $I_k$  are finite for every  $h \geq 0$ , it holds  $\sup_{i_h \in I_h} k_{i_h} \in \{k_{i_h} \mid i_h \in I_h\}$  for every  $h \geq 0$ , hence there exist a sequence  $(i_j)_j \in I_0 \times I_1 \times I_2 \times \dots$  such that  $A = \liminf (k_{i_j})_{j \geq 0}$ , and thus  $A \leq B$ .

Let now  $(i_j)_j \in I_0 \times I_1 \times I_2 \times \dots$ , and  $h \geq 0$  be the maximum  $j \geq 0$  such that  $\{k_{i_h} \mid i_h \in I_h\} \cap L \setminus \{\infty, -\infty\} \neq \emptyset$ ,  $\{k_{i_h} \mid i_h \in I_h\} \cap \{\infty, -\infty\} \neq \emptyset$ , then

$$\begin{aligned}
 \liminf (k_{i_j})_{j \in \mathbb{N}} &= \lim_{j \geq 0} (\inf \{k_{i_l} \mid l \geq j, k_{i_l} \neq \infty\}) \\
 &= \lim_{j > h} (\inf \{k_{i_l} \mid l \geq j, k_{i_l} \neq \infty\}) \\
 &\leq \lim_{j > h} \left( \inf \left\{ \sup_{i_l \in I_l} k_{i_l} \mid l \geq j, \sup_{i_l \in I_l} k_{i_l} \neq \infty \right\} \right) \\
 &= \lim_{j \geq 0} \left( \inf \left\{ \sup_{i_l \in I_l} k_{i_l} \mid l \geq j, \sup_{i_l \in I_l} k_{i_l} \neq \infty \right\} \right) \\
 &= \liminf \left( \sup_{i_j \in I_j} k_{i_j} \right)_{j \in \mathbb{N}} = A
 \end{aligned}$$

where the second and third equality hold because the sequences

$$(\inf \{k_{i_l} \mid l \geq j, k_{i_l} \neq \infty\})_{j \geq 0}, \left( \inf \left\{ \sup_{i_l \in I_l} k_{i_l} \mid l \geq h, \sup_{i_l \in I_l} k_{i_l} \neq \infty \right\} \right)_{j \geq 0}$$

are increasing, and the inequality holds by the fact that

$$\inf \{k_{i_l} \mid l \geq j, k_{i_l} \neq \infty\} \leq \inf \left\{ \sup_{i_l \in I_l} k_{i_l} \mid l \geq j, \sup_{i_l \in I_l} k_{i_l} \neq \infty \right\},$$

for every  $j > h$ . Thus,  $A = B$  as wanted.

We prove now Property 2. We will prove that for all  $k_i (i \geq 1) \in \overline{\mathbb{R}}$ ,  $\liminf(\infty, k_1, k_2, \dots) = \liminf(k_1, k_2, \dots)$ . First we assume that  $\exists i \geq 1$ , such that  $k_i = -\infty$ , then  $\liminf(\infty, k_1, k_2, \dots) = \liminf(k_1, k_2, \dots) = -\infty$ . Otherwise we point out the following cases.

- If  $\forall i \geq 1, k_i = \infty$ , then  $\liminf(\infty, k_1, k_2, \dots) = \liminf(k_1, k_2, \dots) = \infty$ .
- If there exist infinitely many  $i \geq 1$  such that  $k_i \neq \infty$ , then
 
$$\begin{aligned} \liminf(\infty, k_1, k_2, \dots) &= \lim_{j \geq 1} (\inf \{k_{i_j} \mid i_j \geq j, k_{i_j} \neq \infty\}) \\ &= \liminf(k_1, k_2, \dots). \end{aligned}$$
- Finally, if there exist finitely many  $i \geq 1$  such that  $k_i \neq \infty$ , then  $\liminf(\infty, k_1, k_2, \dots) = \inf \{k_{i_j} \mid j \geq 1, k_{i_j} \neq \infty\} = \liminf(k_1, k_2, \dots)$ .

We prove now Property 3. Let  $k \in \overline{\mathbb{R}}$ , then we point out the cases  $k = \infty, k = -\infty$  or  $k \neq \infty, -\infty$ , and we conclude by the definition that  $k = \liminf(k, \infty, \infty, \dots)$  in all three cases, as wanted.

The rest of the properties of generalized product  $\omega$ -valuation monoids are concluded in a straightforward way by the definition of the  $\liminf$ -function, and the operations of  $\sup, \inf$ .  $\square$

**Definition of Procedure 3** Let  $\{\lambda_1, \dots, \lambda_k\}$  be the subset of  $cl(\psi) \cap cl(\xi)$  containing all formulas of the form  $\chi U \varrho$ , and let  $\varphi_1 U \varphi_2 = \lambda_1$ . We let  $\tilde{P}_1 = B_{\tilde{\xi}_0} B_{\tilde{\xi}_{re}^0} B_{\tilde{\xi}_1} B_{\tilde{\xi}_{re}^1} \dots$  where  $B_{\tilde{\xi}_i}, B_{\tilde{\xi}_{re}^i}, i \geq 0$ , are defined in the following way. For all  $i < i_1 - 1$  we set  $\tilde{\xi}^i = \xi^i, B_{\tilde{\xi}_i} = B_{\xi^i}$ , and  $B_{\tilde{\xi}_{re}^i} = B_{\xi_{re}^i}$ . We set  $\tilde{\xi}^{i_1-1} = \xi^{i_1-1}$ , and  $B_{\tilde{\xi}_i^{i_1-1}} = B_{\xi^{i_1-1}}$ , and  $B_{\tilde{\xi}_{re}^{i_1-1}} = B_{\xi_{re}^{i_1-1}} \cup M_{B_{\psi^{i_1-1}, \varphi_1 U \varphi_2}}$ . We have that  $\psi^{i_1} = \beta^{i_1} \wedge (\beta')^{i_1}$  with  $(\beta')^{i_1} \in next(M_{B_{\psi^{i_1-1}, \varphi_1 U \varphi_2}})$ .  $P_w^1$  satisfies the acceptance condition for  $\varphi_1 U \varphi_2$  at position  $i_1$ , i.e.,  $(\beta')^{i_1} \in next(M_{B_{\psi^{i_1-1}, \varphi_2}})$ . Then,  $\psi_{re}^{i_1} = (\beta_{re}^{i_1} \wedge (\beta')_{re}^{i_1})_{re}$ , and let **sequence 1**

$$B_{\beta_{re}^{i_1}} \xrightarrow{\pi_{i_1}} B_{\beta^{i_1+1}} \xrightarrow{\varepsilon} B_{\beta_{re}^{i_1+1}} \xrightarrow{\pi_{i_1+1}} B_{\beta^{i_1+2}} \xrightarrow{\varepsilon} B_{\beta_{re}^{i_1+2}} \rightarrow \dots,$$

and **sequence 2**

$$B_{(\beta')_{re}^{i_1}} \xrightarrow{\pi_{i_1}} B_{(\beta')^{i_1+1}} \xrightarrow{\varepsilon} B_{(\beta')_{re}^{i_1+1}} \xrightarrow{\pi_{i_1+1}} B_{(\beta')^{i_1+2}} \xrightarrow{\varepsilon} B_{(\beta')_{re}^{i_1+2}} \rightarrow \dots$$

be the sequences obtained by Procedure 1. Also,  $\xi^{i_1} = \zeta^{i_1} \wedge (\zeta')^{i_1}$ , where  $(\zeta')^{i_1}$  is the conjunction of elements of  $next\left(M_{B_{\xi^{i_1-1}, \varphi_1 U \varphi_2}}\right) \setminus next\left(M_{B_{\xi^{i_1-1}, \varphi_2}}\right)$  that appear in  $\xi^{i_1}$ . Then,  $\xi_{re}^{i_1} = \left(\zeta_{re}^{i_1} \wedge (\zeta')_{re}^{i_1}\right)_{re}$ , and let **sequence 3**

$$B_{\zeta_{re}^{i_1}} \xrightarrow{\pi_{i_1}} B_{\zeta^{i_1+1}} \xrightarrow{\varepsilon} B_{\zeta_{re}^{i_1+1}} \xrightarrow{\pi_{i_1+1}} B_{\zeta^{i_1+2}} \xrightarrow{\varepsilon} B_{\zeta_{re}^{i_1+2}} \rightarrow \dots,$$

and **sequence 4**

$$B_{(\zeta')_{re}^{i_1}} \xrightarrow{\pi_{i_1}} B_{(\zeta')^{i_1+1}} \xrightarrow{\varepsilon} B_{(\zeta')_{re}^{i_1+1}} \xrightarrow{\pi_{i_1+1}} B_{(\zeta')^{i_1+2}} \xrightarrow{\varepsilon} B_{(\zeta')_{re}^{i_1+2}} \rightarrow \dots$$

be the sequences obtained by Procedure 1. Then, we obtain  $\tilde{\xi}^i, B_{\tilde{\xi}_{re}^i}$  for  $i \in \{i_1 + 1, \dots, i_2 - 1\}$  following procedure 2 for sequences 2, and 3. We apply inductively the construction for all  $i_j, j \geq 2$ . Observe that since  $\varphi_1 U \varphi_2$  does appear in the scope of a next operator, then whenever  $\varphi_1 U \varphi_2$  appears as part of the conjunction of form A of the maximal formula of a state in  $P_w^1, P_w^2$ , then it is obtained from a next formula of the non-empty  $\varphi_1 U \varphi_2$ -consistent subset of the previous state.

For every  $2 \leq m \leq k$  we obtain  $\tilde{P}_m$  by applying the previous procedure for  $\tilde{P}_{m-1}$  and  $P_w^1$ . Then, we set  $\tilde{P}_w^2 = \tilde{P}_k$ .

We present the proof of Lemma 14.

**Proof:** Let  $\psi = \bigvee_{1 \leq l \leq m} (k_l \wedge \psi_l)$  where  $k_l \in K$ , and  $\psi_l \in bLTL(K, AP)$ .

First, we assume that all  $\xi_j$  ( $1 \leq j \leq n$ ) are different from *true* and  $\psi \notin bLTL(K, AP)$ , and we point out the following cases.

(a)  $\xi_j = \bigwedge_{1 \leq i_j \leq m_j} \xi^{(j, i_j)}$  with  $\xi^{(j, i_j)} \in bLTL(K, AP)$  for every  $i_j \in \{1, \dots, m_j\}$ ,

$j \in \{1, \dots, k\}$ , and for every  $j \in \{2, \dots, k\}$  there exist  $i_j^1, \dots, i_j^{h_j} \in \{1, \dots, m_j\}$ , such that for every  $k \in \{1, \dots, h_j\}$ ,  $\xi^{(j, i_j^k)} = \xi^{(j', i_{j'})}$  for some  $j' \in \{1, \dots, j-1\}$ ,  $i_{j'} \in \{1, \dots, m_{j'}\}$ . Then,  $\xi'_j = \bigwedge_{1 \leq i_j \leq m_j} (\xi^{(j, i_j)})'$ ,

where  $(\xi^{(j, i_j)})' \in \widehat{next}\left(M_{B_{\xi_j, \xi^{(j, i_j)}}}\right)$  for every  $i_j \in \{1, \dots, m_j\}$ ,

$j \in \{1, \dots, k\}$ . Moreover, it holds  $\psi \in stLTL(K, AP)$ , and thus  $\psi' \in bLTL(K, AP)$ . Clearly,

$$\varphi = (\psi U \xi) \wedge \left( \left( \bigwedge_{1 \leq i_1 \leq m_1} \xi^{(j, i_1)} \right) \wedge \left( \bigwedge_{2 \leq j \leq k} \left( \bigwedge_{\substack{1 \leq i_j \leq m_j \\ i_j \neq i_j^1, \dots, i_j \neq i_j^{h_j}}} \xi^{(j, i_j)} \right) \right) \right).$$

Let  $B_\varphi = \{\psi U \xi\} \cup B_\psi \cup \left( \bigcup_{1 \leq j \leq k} \left( \bigcup_{1 \leq i_j \leq m_j} M_{B_{\xi_j, \xi^{(j, i_j)}}} \right) \right)$ . We can prove that  $B_\varphi$  is a  $\varphi$ -consistent set following the arguments of proof of Lemma 107 in [21]. Moreover,  $M_{B_{\xi_j, \xi^{(j, i_j)}}} \subseteq M_{B_\varphi, \xi^{(j, i_j)}} (1 \leq j \leq k, 1 \leq i_j \leq m_j)$  and  $B_\psi \subseteq M_{B_\varphi, \psi}$ . Then, with the same arguments used

in Lemma 10, we get that  $\varphi' = (\psi U \xi) \wedge \psi' \wedge \left( \bigwedge_{1 \leq i_1 \leq m_1} (\xi^{(1, i_1)})' \right) \wedge \left( \bigwedge_{2 \leq j \leq k} \left( \bigwedge_{\substack{1 \leq i_j \leq m_j \\ i_j \neq i_j^1, \dots, i_j \neq i_j^{h_j}}} (\xi^{(j, i_j)})' \right) \right) \in \widehat{next}(B_\xi)$ . Therefore,  $(B_\varphi, \pi, B_{\varphi'})$  is a next transition and

$$\begin{aligned} v_{B_\varphi}(\varphi') &= v_{B_\psi}(\psi') \cdot \prod_{1 \leq i_1 \leq m_1} v_{M_{B_{\xi_j, \xi^{(1, i_1)}}}} \left( \left( (\xi^{(1, i_1)})' \right) \right) \\ &\quad \cdot \left( \prod_{2 \leq j \leq k} \left( \prod_{\substack{1 \leq i_j \leq m_j \\ i_j \neq i_j^1, \dots, i_j \neq i_j^{h_j}}} \left( v_{M_{B_{\xi_j, \xi^{(j, i_j)}}}} \left( \left( (\xi^{(j, i_j)})' \right) \right) \right) \right) \right) \\ &\geq v_{B_\psi}(\psi') \cdot \prod_{1 \leq i_1 \leq m_1} v_{M_{B_{\xi_j, \xi^{(1, i_1)}}}} \left( \left( (\xi^{(1, i_1)})' \right) \right) \end{aligned}$$

$$\begin{aligned}
 & \cdot \left( \prod_{2 \leq j \leq k} \left( \prod_{\substack{1 \leq i_j \leq m_j \\ i_j \neq i_j^1, \dots, i_j \neq i_j^{h_j}}} \left( v_{M_B} \right)_{\xi_j, \xi^{(j, i_j)}} \left( \left( \xi^{(j, i_j)} \right)' \right) \right) \right) \\
 & \cdot \prod_{2 \leq j \leq k} \left( \prod_{i_j \in \{i_j^1, \dots, i_j^{h_j}\}} \left( v_{M_B} \right)_{\xi_j, \xi^{(j, i_j)}} \left( \left( \xi^{(j, i_j)} \right)' \right) \right) \\
 & = v_{B_\psi} (\psi') \cdot \prod_{1 \leq j \leq k} v_{B_{\xi_j}} (\xi'_j)
 \end{aligned}$$

where the inequality is obtained using Lemma 10, Remark 1, and the same arguments that are used in the proof of the corresponding inequality of Lemma 12.

We have completed the proof of (i). We prove now (ii). The claim of (ii) trivially holds for  $\psi'' = \psi'$ ,  $\xi''_1 = \xi'_1$ . For  $2 \leq j \leq k$  we set

$$\xi''_j = \bigwedge_{\substack{1 \leq i_j \leq m_j \\ i_j \neq i_j^1, \dots, i_j \neq i_j^{h_j}}} (\xi^{(j, i_j)})'. \text{ It holds}$$

$$\begin{aligned}
 \varphi' & = (\psi U \xi) \wedge \psi' \wedge \left( \bigwedge_{1 \leq i_1 \leq m_1} (\xi^{(1, i_1)})' \right) \\
 & \wedge \left( \bigwedge_{2 \leq j \leq k} \left( \bigwedge_{\substack{1 \leq i_j \leq m_j \\ i_j \neq i_j^1, \dots, i_j \neq i_j^{h_j}}} (\xi^{(j, i_j)})' \right) \right),
 \end{aligned}$$

and

$$\varphi'_{re} = \left( (\psi U \xi) \wedge \psi''_{re} \wedge (\xi''_1)_{re} \wedge \left( \bigwedge_{2 \leq j \leq k} (\xi''_j)_{re} \right) \right)_{re},$$

Let now  $j \in \{2, \dots, k\}$ . We consider now the infinite sequence of next and  $\varepsilon$ -reduction transitions

$$B_{\psi^0} \xrightarrow{\pi_0} B_{\psi^1} \xrightarrow{\varepsilon} B_{\psi^1_{re}} \xrightarrow{\pi_1} B_{\psi^2} \xrightarrow{\varepsilon} B_{\psi^2_{re}} \dots$$

with  $\psi^0 = (\xi'_j)_{re}$  and  $v_{B_{\psi^i_{re}}}(\psi^{i+1}) \neq \mathbf{0}$  ( $i \geq 0$ ). Clearly,

$$\psi^0 = (\xi'_j)_{re} = \left( (\xi''_j)_{re} \wedge \left( (\xi^{(j,i_1^j)})' \wedge \dots \wedge (\xi^{(j,i_{h_j^j})})' \right) \right)_{re}.$$

Then, for  $\lambda^0 = (\xi''_j)_{re}$ , and  $\zeta^0 = \left( (\xi^{(j,i_1^j)})' \wedge \dots \wedge (\xi^{(j,i_{h_j^j})})' \right)_{re}$ ,

by induction on  $i$  and Lemma 11, we obtain that for every  $i \geq 0$ , there exist a  $\lambda^i_{re}$ -consistent set  $B_{\lambda^i_{re}}$ , and a  $\zeta^i_{re}$ -consistent set  $B_{\zeta^i_{re}}$ , and formulas  $\lambda^{i+1} \in \text{next}(B_{\lambda^i_{re}})$ ,  $\zeta^{i+1} \in \text{next}(B_{\zeta^i_{re}})$  such that

$$\psi^{i+1}_{re} = (\lambda^{i+1}_{re} \wedge \zeta^{i+1}_{re})_{re},$$

and

$$v_{B_{\psi^i_{re}}}(\psi^{i+1}) = v_{B_{\lambda^i_{re}}}(\lambda^{i+1}) \cdot v_{B_{\zeta^i_{re}}}(\zeta^{i+1}).$$

For every  $i \geq 0$ ,  $v_{B_{\psi^i_{re}}}(\psi^{i+1}) \neq \mathbf{0}$  and  $\zeta^i_{re}$  is boolean, hence  $v_{B_{\zeta^i_{re}}}(\zeta^{i+1}) = \mathbf{1}$ , i.e.,  $v_{B_{\psi^i_{re}}}(\psi^{i+1}) = v_{B_{\lambda^i_{re}}}(\lambda^{i+1})$  for every  $i \geq 0$ . So, the sequence

$$B_{\lambda^0} \xrightarrow{\pi_0} B_{\lambda^1} \xrightarrow{*} B_{\lambda^1_{re}} \xrightarrow{\pi_1} B_{\lambda^2} \xrightarrow{*} B_{\lambda^2_{re}} \dots$$

satisfies the lemma's claim.

- (b) If  $\left( (\psi U \xi) \wedge \left( \bigwedge_{1 \leq j \leq k} \xi_j \right) \right)_{re} = (\psi U \xi) \wedge \left( \bigwedge_{1 \leq j \leq k} \xi_j \right)$ , we set  $B_\varphi = \{\psi U \xi\} \cup B_\psi \cup \left( \bigcup_{1 \leq j \leq k} B_{\xi_j} \right)$ , and we proceed in the same way.

Finally, we use the same arguments to prove our claim in the cases where at least one of  $\xi_j$  ( $1 \leq j \leq k$ ) equals to *true*.

□