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On Quantum-MV Algebras - Part I: The Orthomodular Algebras

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In memoriam Dragoș Vaida (1933 - 2020)

Abstract

We prove that almost all the properties of quantum-MV algebras are verified by orthomodular algebras, the new algebras introduced in a previous paper. We put a special insight on transitive antisymmetric orthomodular (taOM) algebras, generalizations of MV algebras. We make the connection with IMTL and NM algebras.

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Keywords: m-MEL algebra, m-BE algebra, m-aBE algebra, m-pre-BCK algebra, m-BCK algebra, m-BCK lattice, IMTL algebra, NM algebra, MV algebra, $_{(WNM)}$MV algebra, quantum-MV algebra, orthomodular algebra, taOM algebra, Boolean algebra

1 Introduction

The *algebraists* work usually with the commutative additive groups and with the positive (right) cone of a partially ordered commutative group $(G, \leq, +, -, 0)$, where there are essentially a sum $\oplus = +$ and an element 0. Sometimes, the negative (left) cone is needed also, where there are essentially

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a product $\odot = +$ and an element $1 = 0$. They work with algebras that have associated an (pre-order) order relation, which usually does not appear explicitely in the definitions. The presence of the (pre-order) order relation implies the presence of the (generalized) duality principle. Thus, each algebra has a dual one, the (pre-order) order relation has a dual one. We have given names to the dual algebras [17, 20, 22]: "left" algebra and "right" algebra, names connected with the left-continuity of a t-norm and with the right-continuity of a t-conorm, respectively. Hence, the algebraists usually work with the commutative *right-unital magmas*.

By contrary, the *logicians* work with the logic of *truth*, where the *truth* is represented by 1, and there is essentially one implication; we could name this logic "left-logic". One can imagine also a "right-logic", as a logic of *false*, where the *false* is represented by 0 and there is a "right-implication". Hence, the logicians usually work with the commutative *left-algebras of logic*.

In this paper, *regarding from (algebras of) logic side, we shall work with left-algebras (left-unital magmas), in principal*, therefore, the unital magmas will be defined multiplicatively, in principal.

Thus, the commutative algebraic structures connected directly or indirectly with classical/ nonclassical logics belong to two parallel "worlds":

1. the "world" of *(left) algebras of logic*, where there are essentially one implication, $\to$ (two, in the non-commutative case), and an element 1 (that can be the last element); the algebras $(A, \to, 1)$, verifying the basic property (M): $1 \to x = x$, are called *M algebras* [20, 21, 22]; an internal binary relation can be defined by: $x \leq y \overset{def.}{\Longleftrightarrow} x \to y = 1$ ($\leq$ can be a pre-order, an order, or even a lattice order); algebras belonging to this "world" are [18, 19, 20, 21, 22]: the bounded MEL, BE and aBE, pre-BCK algebras, BCK algebras, bounded BCK algebras, BCK(P) algebras, Hilbert algebras, Wajsberg algebras, implicative-Boolean algebras, etc. A "Big map" (hierarchy of algebras of logic) is presented in ([22], Figure 1).

2. the "world" of *(left) algebras*, where there are essentially a product, $\odot$, and an element 1 (that can be the last element); the algebras $(A, \odot, 1)$, verifying the corresponding basic properties (PU): $1 \odot x = x$ and (Pcomm): $x \odot y = y \odot x$, are called *commutative unital magmas*; an internal binary relation can be defined, if an additional negation, $^-$, exists and $0 \overset{def.}{=} 1^-$, by: $x \leq_m y \Longleftrightarrow x \odot y^- = 0$ ($\leq_m$ can be a pre-order, an order, or even a lattice order), where 'm' comes from

'magma'; algebras belonging to this "world" are [20, 22]: the m-MEL, m-BE and m-aBE, m-pre-BCK algebras, m-BCK algebras, pocrims, (bounded) lattices, residuated lattices, BL algebras, MTL algebras, NM algebras, MV algebras, Boolean algebras, etc. A corresponding "Big map" (hierarchy of algebras) is presented in ([22], Figure 10).

Between the two parallel " worlds" there are some connections, as for examples: the equivalence between BCK(P) algebras and pocrims, in the non-involutive case, and the definitional equivalence between Wajsberg algebras and MV algebras, in the involutive case ($(x^-)^- = x$).

Beside the classical and non-classical logics, there exist the quantum logics. Examples of algebraic structures connected with quantum logics (= quantum structures/ algebras) are: the bounded implicative (implication) lattices, the De Morgan algebras, the ortholattices, the orthomodular lattices, the quantum-MV algebras, etc.

Quantum-MV algebras (or QMV algebras) were introduced by Roberto Giuntini in [10] (see also [12, 9, 11, 13, 15, 14, 16, 5]), as non-lattice theoretic generalizations of MV algebras [2, 4] and as non-idempotent generalizations of orthomodular lattices [31, 28].

The connections between algebras of logic/ algebras and quantum algebras were not very clear. But, in papers [22, 25, 27], we established important connections, by redefining equivalently the bounded involutive lattices and the De Morgan algebras [3] as involutive *m-MEL algebras*, the ortholattices [3], the MV, the Boolean algebras and the quantum-MV algebras as involutive *m-BE algebras*, verifying some properties, and then putting all of them on the involutive "Big map"; thus, we have proved that the quantum algebras belong, in fact, to the "world" of *algebras* (involutive commutative unital magmas).

In this paper, we continue the research from [25, 26, 27] based on [22], in the "world" of involutive *algebras* of the form $(A, \odot, ^-, 1)$, with $1^- = 0$, 1 *being the last element*: we analyse in some details the orthomodular (OM) algebras defined in [27], with a special insight on taOM algebras.

The paper is organized as follows. In Section 2 (**Preliminaries**), we recall from [22, 25, 27] the necessary definitions and results which make this paper selfcontained as much as possible. In Section 3 (**Orthomodular algebras**), we prove that almost all the properties of QMV algebras are also verified by orthomodular (OM) algebras; we put OM algebras on the "map". In Section 4 (**A new algebra: the trans algebra (TRANS)**), we mainly prove that any m-BCK algebra verifies the property (trans) (the binary relation $\leq_m^M$ is transitive) and we introduce and analyse the so called *trans algebras*. In Section 5 (**The taOM algebras inside the m-BCK algebras**), we prove the definitional equivalence between involutive residuated lattices and m-BCK lattices, thus putting IMTL, NM, MV and $_{(WNM)}$MV algebras on the same "map". Concerning the taOM algebras, all the finite examples we found are m-BCK lattices and the open problem is if any taOM algebra is an m-BCK lattice. In Section 6 (**Examples**), we present 18 examples of the involved algebras.

This paper, like [22, 25, 26, 27], presents the facts in the same *unifying way*, which consists in fixing unique names for the defining properties, making lists of these properties and then using them for defining the different algebras and for obtaining results.

2 Preliminaries

2.1 The "Big map" of Algebras

Recall from [22] the following:

Let $\mathcal{A}^L = (A^L, \odot, ^- = ^{-^L}, 1)$ be an algebra of type $(2, 1, 0)$ and define $0 \stackrel{def.}{=} 1^-$. Define an *internal* binary relation $\leq_m$ on A^L by: for all $x, y \in A^L$,

(m-dfrelP) $\quad x \leq_m y \stackrel{def.}{\Longleftrightarrow} x \odot y^- = 0.$

Consider the following list **m-A** of basic properties that can be satisfied by $\mathcal{A}^L$ [22]:

(PU) $\quad 1 \odot x = x = x \odot 1$ (unit element of product, the *identity*),
(Pcomm) $\quad x \odot y = y \odot x$ (commutativity of product),
(Pass) $\quad x \odot (y \odot z) = (x \odot y) \odot z$ (associativity of product);
(Neg1-0) $\quad 1^- = 0$,
(Neg0-1) $\quad 0^- = 1$;
(m-An) $\quad (x \odot y^- = 0$ and $y \odot x^- = 0) \Longrightarrow x = y$ (antisymmetry),
(m-B) $\quad [(x \odot y^-)^- \odot (x \odot z)] \odot (y \odot z)^- = 0$,
(m-BB) $\quad [(z \odot x)^- \odot (y \odot x)] \odot (y \odot z^-)^- = 0$,
(m-*) $\quad x \odot y^- = 0 \Longrightarrow (z \odot y^-) \odot (z \odot x^-)^- = 0$,
(m-**) $\quad x \odot y^- = 0 \Longrightarrow (x \odot z) \odot (y \odot z)^- = 0$,
(m-L) $\quad x \odot 0 = 0$ (last element),
(m-Re) $\quad x \odot x^- = 0$ (reflexivity),
(m-Tr) $\quad (x \odot y^- = 0$ and $y \odot z^- = 0) \Longrightarrow x \odot z^- = 0$ (transitivity),
etc.

Dually, let $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-^R}, 0)$ be an algebra of type $(2,1,0)$ and define $1 \stackrel{def.}{=} 0^-$. Define an *internal* binary relation $\geq_m$ on A^R by: for all $x, y \in A^R$,

(m-dfrelS) $\quad x \geq_m y \stackrel{def.}{\Longleftrightarrow} x \oplus y^- = 1.$

The list of dual properties is omitted.

Recall from [22] the definitions of the algebras needed in this paper (the dual ones are omitted):

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an algebra of type $(2,1,0)$ through this paper. Define $0 \stackrel{def.}{=} 1^-$ (hence (Neg1-0) holds) and suppose that $0^- = 1$ (hence (Neg0-1) holds too). We say that $\mathcal{A}^L$ is a [22]:

- *left-m-MEL algebra*, if (PU), (Pcomm), (Pass), (m-L) hold;
- *left-m-BE algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) hold;
- *left-m-pre-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re) and (m-BB) hold;
- *left-m-BCK algebra*, if (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An) and (m-BB) hold.

Denote by **m-MEL**, **m-BE**, **m-pre-BCK**, **m-BCK** these classes of left-algebras, respectively.

In ([22], Figure 10), the "Big map", connecting the commutative unital magmas, including these algebras, was drawn.

We say that $\mathcal{A}^L$ is [22] *reflexive*, if $\leq_m$ is reflexive (i.e. (m-Re) holds); *transitive*, if $\leq_m$ is transitive (i.e. (m-Tr) holds); *antisymmetric*, if $\leq_m$ is antisymmetric (i.e. (m-An) holds). If **X** is a class of algebras, we shall denote by **tX** (**aX**, **atX**=**taX**) the subclass of all transitive (antisymmetric, transitive and antisymmetric, respectively) algebras of **X**.

We say that an algebra is *involutive*, if it verifies (DN) (Double Negation) ($(x^-)^- = x$ or $x^= = x$). If **X** is a class of algebras, we shall denote by $\mathbf{X}_{(DN)}$ the subclass of all involutive algebras of **X**. By ([22], Theorem 6.12), in any involutive m-BE algebra we have the equivalences: (m-BB) $\Leftrightarrow$ (m-B) $\Leftrightarrow$ (m-**) $\Leftrightarrow$ (m-*) $\Leftrightarrow$ (m-Tr).

Note that: $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)} = \mathbf{pre\text{-}m\text{-}BCK}_{(DN)}$ ($= \mathbf{m\text{-}tBE}_{(DN)}$).

Any left-m-BCK algebra is involutive, by ([22], Theorem 6.13). We write: $\mathbf{m\text{-}BCK} = \mathbf{m\text{-}BCK}_{(DN)}$ ($= \mathbf{m\text{-}taBE}_{(DN)}$). Note that an (involutive) m-BCK algebra satisfies all the properties in the list **m-A** of properties and, additionally, (DN) and other properties.

Note that the binary relation $\leq_m$ is only **reflexive** in $\mathbf{m\text{-}BE}_{(DN)}$, it is a **pre-order** in $\mathbf{m\text{-}pre\text{-}BCK}_{(DN)}$ and it is an **order** in **m-BCK**.

2.1.1 Involutive m-MEL Algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. Because of the axiom (DN), we have introduced in [25] the new operation sum, $\oplus$, the dual of product, $\odot$, by: for all $x, y \in A^L$,

$$x \oplus y \stackrel{def.}{=} (x^- \odot y^-)^-. \tag{1}$$

Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-MEL algebra.

Proposition 2.1 *(See ([5], Proposition 2.1.2), in dual case, [12])*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:

$$0 \oplus x = x = x \oplus 0, \quad i.e. \quad (SU) \quad holds, \tag{2}$$

$$x \oplus y = y \oplus x, \quad i.e. \quad (Scomm) \quad holds, \tag{3}$$

$$x \oplus (y \oplus z) = (x \oplus y) \oplus z, \quad i.e. \quad (Sass) \quad holds, \tag{4}$$

$$x \oplus 1 = 1, \quad i.e. \quad (m - L^R) \quad holds; \tag{5}$$

$$(x \oplus y)^- = x^- \odot y^- \quad (De\ Morgan\ law\ 1), \tag{6}$$

$$(x \odot y)^- = x^- \oplus y^- \quad (De\ Morgan\ law\ 2),\ and\ hence \tag{7}$$

$$x \odot y = (x^- \oplus y^-)^-. \tag{8}$$

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [25] a new binary relation:

(m-dfP) $x \ \leq_m^P \ y \stackrel{def.}{\Longleftrightarrow} x \odot y = x$ and, dually,

(m-dfS) $x \ \geq_m^S \ y \stackrel{def.}{\Longleftrightarrow} x \oplus y = x$.

By ([25], Proposition 3.11), $\leq_m^P$ is antisymmetric and transitive and $0 \leq_m^P x \leq_m^P 1$, for any x, where $0 \stackrel{def.}{=} 1^-$.

With the notations from this subsection, the definition of MV algebras becomes [22]:

Definition 2.2 (i) A *left-MV algebra* is an algebra $\mathcal{A}^L$=$(A^L, \odot, ^- = ^{-^L}$,1) of type $(2, 1, 0)$ verifying (PU), (Pcomm), (Pass), (m-L), (DN) and:

($\wedge_m$-comm) $(x^- \odot y)^- \odot y = (y^- \odot x)^- \odot x$.

(i') Dually, a *right-MV algebra* is an algebra $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-R}, 0)$ of type $(2, 1, 0)$ verifying (SU), (Scomm), (Sass), (m-L^R), (DN) and: ($\vee_m$-comm) $(x^- \oplus y)^- \oplus y = (y^- \oplus x)^- \oplus x$.

We recall the following important remark, which was the motivation of paper [22]:

(i) The left-MV algebra is just the involutive left-m-MEL algebra verifying ($\wedge_m$-comm).

(i') Dually, the right-MV algebra is just the involutive right-m-MEL algebra verifying ($\vee_m$-comm).

Denote by **MV** the class of all left-MV algebras and by $\mathbf{MV}^R$ the class of all right-MV algebras.

2.1.2 Involutive m-BE Algebras

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, $(A^L, \oplus, ^-, 0)$ is an involutive right-m-BE algebra.

Remark 2.3 (see ([22], Theorem 6.21)) (The dual one is omitted)

Since ($\wedge_m$-comm) implies (m-Re), by ([22], (mB1)), it follows that **any left-MV algebra is in fact an involutive left-m-BE algebra verifying ($\wedge_m$-comm)**. And since ($\wedge_m$-comm) implies also (m-An) and (m-BB) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-Tr)), by ([22], (mB2), (mCBN1)), respectively, it follows that **any left-MV algebra is in fact a left-m-BCK algebra**, i.e. we have:

$$\mathbf{MV} \subset \mathbf{m-BCK} = \mathbf{m-BCK_{(DN)}} \ (= \mathbf{m-taBE_{(DN)}}).$$

In ([22], Figure 8), the connections between m-BE algebras, m-BCK algebras, MV algebras, ortholattices and Boolean algebras were established, thus putting MV algebras, ortholattices and Boolean algebras on the "map" (the right side of the involutive "Big map").

2.2 The Redefined QMV Algebras

We have introduced in [27], in an involutive left-m-MEL algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$, the following new operations:

$$x \wedge_m^M y \stackrel{def.}{=} (x^- \odot y)^- \odot y \stackrel{(Pcomm)}{=} y \odot (y \odot x^-)^- \quad and,\ dually, \quad (9)$$

$$x \vee_m^M y \overset{def.}{=} (x^- \wedge_m^M y^-)^- = [(x \odot y^-)^- \odot y^-]^- = (x \odot y^-) \oplus y \\ = (x^- \oplus y)^- \oplus y = y \oplus (y \oplus x^-)^- \tag{10}$$

and

$$x \wedge_m^B y \overset{def.}{=} (y^- \odot x)^- \odot x \overset{(Pcomm)}{=} x \odot (x \odot y^-)^- = y \wedge_m^M x \quad \textit{and, dually,} \tag{11}$$

$$x \vee_m^B y \overset{def.}{=} (x^- \wedge_m^B y^-)^- = ((y \odot x^-)^- \odot x^-)^- = (y \odot x^-) \oplus x \\ = (y^- \oplus x)^- \oplus x = x \oplus (x \oplus y^-)^- = y \vee_m^M x. \tag{12}$$

In what follows, we shall present only the properties of $\wedge_m^M$ and $\vee_m^M$.

Proposition 2.4 *(See [5], Proposition 2.1.2, in dual case) ([27], Proposition 3.2)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-MEL algebra. We have:

$$x \wedge_m^M 1 = x = 1 \wedge_m^M x, \quad x \wedge_m^M 0 = 0, \tag{13}$$

$$x \vee_m^M 0 = x = 0 \vee_m^M x, \quad x \vee_m^M 1 = 1, \tag{14}$$

$$(x \vee_m^M y)^- = x^- \wedge_m^M y^- \quad (\textit{De Morgan law } 1), \tag{15}$$

$$(x \wedge_m^M y)^- = x^- \vee_m^M y^- \quad (\textit{De Morgan law } 2), \textit{ and hence} \tag{16}$$

$$x \wedge_m^M y = (x^- \vee_m^M y^-)^-. \tag{17}$$

Proposition 2.5 *(See ([5], Proposition 2.1.2), in dual case) ([27], Proposition 3.3)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:

$$if \quad x \odot y = 1, \quad then \quad x = y = 1; \tag{18}$$

$$if \quad x \wedge_m^M y = 1, \quad then \quad x = y = 1, \tag{19}$$

$$0 \wedge_m^M x = 0, \tag{20}$$

$$1 \vee_m^M x = 1, \tag{21}$$

$$x \wedge_m^M x = x, \quad x \vee_m^M x = x, \tag{22}$$

$$if \quad x \leq_m^M y, \quad then \quad y \wedge_m^M x = x; \tag{23}$$

$$if \quad x \leq_m^M y, \quad then \quad x \leq_m y. \tag{24}$$

Proposition 2.6 *([27], Proposition 3.4)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:

$$x \oplus x^- = 1, \quad i.e. \quad (m - Re^R) \quad holds; \tag{25}$$

$$x \odot (y \wedge_m^M x^-) = 0, \tag{26}$$

$$x \odot (x^- \wedge_m^M y) = 0, \tag{27}$$

$$(y \vee_m^M x) \wedge_m^M x = x, \tag{28}$$

$$(y \wedge_m^M x) \vee_m^M x = x, \tag{29}$$

$$if \quad x \leq_m^M y, \quad then \quad x \vee_m^M y = y, \tag{30}$$

$$x \vee_m^M y = y \iff x \odot y^- = 0 \quad (\iff x \leq_m y), \tag{31}$$

$$(x \odot y) \vee_m^M x = x, \tag{32}$$

$$x \wedge_m^M (x \odot y) = x \odot y, \tag{33}$$

$$x \wedge_m^M (y \wedge_m^M x) = y \wedge_m^M x. \tag{34}$$

Beside the old, natural binary relation $\leq_m$ and its dual $\geq_m$, we have introduced in [27] two new binary relations: for all $x, y \in A^L$,

(m-dfWM) $x \leq_m^M y \stackrel{def.}{\iff} x \wedge_m^M y = x$ and, dually,

(m-dfVM) $x \geq_m^M y \stackrel{def.}{\iff} x \vee_m^M y = x$,

and

(m-dfWB) $x \leq_m^B y \stackrel{def.}{\iff} x \wedge_m^B y = x$ $(\iff y \wedge_m^M x = x)$ and, dually,

(m-dfVB) $x \geq_m^B y \stackrel{def.}{\iff} x \vee_m^B y = x$ $(\iff y \vee_m^M x = x)$.

Proposition 2.7 *([27], Proposition 3.6)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. We have:

(1) $x \leq_m y \iff x \leq_m^B y$ *and, dually*

(1') $x \geq_m y \iff x \geq_m^B y$.

(2) *If ($\wedge_m$-comm) holds (i.e.* $x \wedge_m^M y = y \wedge_m^M x$*), then* $x \leq_m y$ $(\iff x \leq_m^B y)$ $\iff x \leq_m^M y$.

(2') *If ($\wedge_m$-comm) holds, then ($\vee_m$-comm) holds (i.e.* $x \vee_m^M y = y \vee_m^M x$*) and* $x \geq_m y$ $(\iff x \geq_m^B y)$ $\iff x \geq_m^M y$.

Remark 2.8 ([27], Remark 3.7)

The equivalence $\leq_m \Longleftrightarrow \leq_m^B$ implies that $\leq_m$ is an order relation if and only if $\leq_m^B$ is an order relation. But, it does not imply that if $\leq_m$ is a lattice order, then $\leq_m^B$ is a lattice order too with respect to $\wedge_m^B, \vee_m^B$ - see the examples in the last section.

Proposition 2.9 *([27], Proposition 3.8)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,

$$(x \leq_m^B y \Longleftrightarrow)\ x \leq_m y \Longleftrightarrow y \geq_m x\ (\Longleftrightarrow y \geq_m^B x).$$

Corollary 2.10 *(See [5], Corollary 2.1.3) (See [27], Corollary 3.9)*

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then, the binary relation $\leq_m^M$ is reflexive and antisymmetric and $0 \leq_m^M x \leq_m^M 1$, for all $x \in A^L$, where $0 \stackrel{def.}{=} 1^-$.

Definitions 2.11 R. Giuntini ([27], Definitions 3.10)

(i) A *left-quantum-MV algebra*, or a *left-QMV algebra* for short, is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^- = ^{-_L}, 1)$ verifying the following axiom: for all $x, y, z \in A^L$,

(Pqmv) $x \odot [(x^- \vee_m^M y) \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$.

(i') A *right-quantum-MV algebra*, or a *right-QMV algebra* for short, is an involutive right-m-BE algebra (= S algebra) $\mathcal{A}^R = (A^R, \oplus, ^- = ^{-_R}, 0)$ verifying the following dual axiom: for all $x, y, z \in A^R$,

(Sqmv) $x \oplus [(x^- \wedge_m^M y) \wedge_m^M (z \wedge_m^M x^-)] = (x \oplus y) \wedge_m^M (x \oplus z)$.

We shall denote by **QMV** the class of all left-QMV algebras and by $\mathbf{QMV}^R$ the class of all right-QMV algebras.

Corollary 2.12 ([27], Corollary 3.12)

Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Then, $(A^L, \oplus, ^-, 0)$ is a right-QMV algebra.

Consider the properties:

(Pom) $(x \odot y) \oplus ((x \odot y)^- \odot x) = x$ or, equivalently, $x \vee_m^M (x \odot y) = x$ and, dually,

(Som) $(x \oplus y) \odot ((x \oplus y)^- \oplus x) = x$ or, equivalently, $x \wedge_m^M (x \oplus y) = x$;

(Pmv) $x \odot (x^- \vee_m^M y) = x \odot y$ and, dually,

(Smv) $x \oplus (x^- \wedge_m^M y) = x \oplus y$;

(Δ_m) $\quad (x \wedge_m^M y) \odot (y \wedge_m^M x)^- = 0$ and, dually,
(∇_m) $\quad (x \vee_m^M y) \oplus (y \vee_m^M x)^- = 1.$

Theorem 2.13 *([27])*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,

(1) $(Pqmv) \iff (Pmv) + (Pom)$,

(2) $(Pmv) \implies (\Delta_m)$,

(3) $(Pom) + (\Delta_m) \implies (Pmv)$,

(4) $(Pqmv) \iff (\Delta_m) + (Pom)$.

We have introduced in [27] the following definitions:

Definitions 2.14 (The dual ones are omitted)
An involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ is:

- a *left-orthomodular algebra*, or a *left-OM algebra* for short, if it verifies (Pom),
- a *left-pre-MV algebra*, or a *left-PreMV algebra* for short, if it verifies (Pmv),
- a *left-metha-MV algebra*, or a *left-MMV algebra* for short, if it verifies (Δ_m).

We have denoted by **OM**, **PreMV**, **MMV** the classes of the corresponding left-algebras.

The connections between all these algebras were established in [27] - see the Figures 1 and 2.

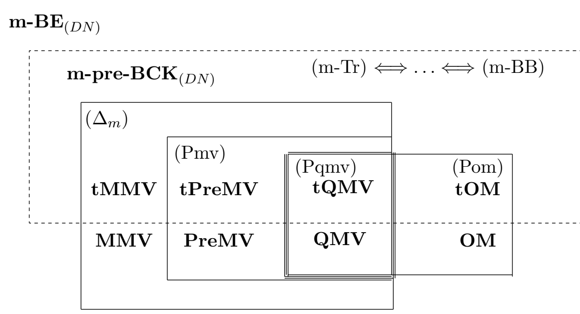


Figure 1: Resuming connections between **OM**, **PreMV**, **MMV**, **QMV** and (m-Tr)

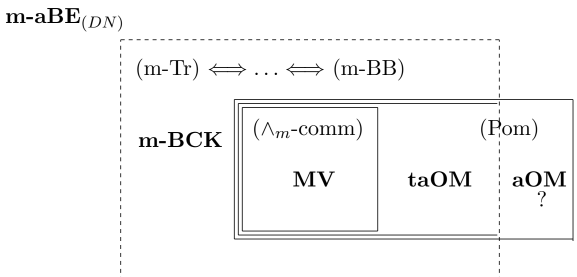


Figure 2: Resuming connections between **MV**, **taOM** and **aOM**, where ? means that there is an open problem concerning **aOM**

We have ([27], Corollary 4.10, Remark 4.11): **aPreMV** = **aMMV** = **aQMV** = **taQMV** = **MV**. Note that *taOM algebras* are proper generalizations of MV algebras inside the class of m-BCK algebras.

Remark 2.15 (see [27]) In a left-MV algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:

- the initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is an **order relation**, since (m-Re), (m-An) and (m-Tr) hold;
- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an **order relation**;
- both $\leq_m$ and $\leq_m^M$ are distributive lattice order relations and they are equivalent: $\leq_m$ ($\Longleftrightarrow \leq_m^B$) $\Longleftrightarrow \leq_m^M$, by Proposition 2.7;
- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is **only antisymmetric and transitive**, by ([25], Proposition 3.11); in Boolean algebras, it is a distributive lattice order and $\leq_m^P \Longleftrightarrow \leq_m \Longleftrightarrow \leq_m^M$.

3 Orthomodular Algebras

3.1 Properties of OM Algebras

We shall see that almost all the properties verified by a QMV algebra are also verified by an OM algebra.

Proposition 3.1 *(See [27], Proposition 3.15 for QMV algebras)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. We have:

$$x \odot (y \vee_m^M x^-) = x \odot y, \tag{35}$$

$$x \odot y \ \leq_m^M \ x, \quad i.e. \quad (x \odot y) \wedge_m^M x = x \odot y, \tag{36}$$

$$x \ \leq_m^M \ x \oplus y, \quad i.e. \quad x \wedge_m^M (x \oplus y) = x, \tag{37}$$

$$x \wedge_m^M y \ \leq_m^M \ y, \quad i.e. \quad (x \wedge_m^M y) \wedge_m^M y = x \wedge_m^M y, \tag{38}$$

$$y \ \leq_m^M \ x \vee_m^M y, \quad i.e. \quad y \wedge_m^M (x \vee_m^M y) = y, \tag{39}$$

$$x \vee_m^M (y \wedge_m^M x) = x, \tag{40}$$

$$x \ \leq_m^M \ y \Longrightarrow y \vee_m^M x = y, \tag{41}$$

$$x \ \leq_m^M \ y \Longrightarrow y^- \ \leq_m^M \ x^- \quad (order-reversibility\ of\ ^-), \tag{42}$$

$$x \ \leq_m^M \ y \Longrightarrow x \oplus z \ \leq_m^M \ y \oplus z \quad (monotonicity\ of\ \oplus), \tag{43}$$

$$x \ \leq_m^M \ y \Longrightarrow x \odot z \ \leq_m^M \ y \odot z \quad (monotonicity\ of\ \odot), \tag{44}$$

$$(x \wedge_m^M y) \wedge_m^M z = (x \wedge_m^M y) \wedge_m^M (y \wedge_m^M z), \tag{45}$$

$$(x \vee_m^M y) \vee_m^M z = (x \vee_m^M y) \vee_m^M (y \vee_m^M z). \tag{46}$$

Proof:

(35) $x \odot (y \vee_m^M x^-)$

$$\begin{aligned}
&\overset{(DN),(10)}{=} && x \odot [(y \odot x)^- \odot x]^- \\
&\overset{(8)}{=} && (x^- \oplus [(y \odot x)^- \odot x])^- \\
&\overset{(Pom)}{=} && ([(x \odot y) \oplus ((x \odot y)^- \odot x)]^- \oplus [(x \odot y)^- \odot x])^- \\
&\overset{(6)}{=} && ([(x \odot y)^- \odot ((x \odot y)^- \odot x)^-] \oplus ((x \odot y)^- \odot x))^- \\
& && \text{(put } X = (x \odot y)^- \text{ and } Y = x) \\
&= && ((X \odot Y) \oplus [(X \odot Y)^- \odot X])^- \\
&\overset{(Pom)}{=} && X^- = ((x \odot y)^-)^- \overset{(DN)}{=} x \odot y.
\end{aligned}$$

(36) $x \odot y$

$$\begin{aligned}
&\overset{(35)}{=} && x \odot (y \vee_m^M x^-) \\
&\overset{(10)}{=} && x \odot [(y \odot x) \oplus x^-] \\
&\overset{(Pcomm)}{=} && x \odot [(x \odot y) \oplus x^-] \\
&\overset{(1)}{=} && x \odot ((x \odot y)^- \odot x)^- \\
&\overset{(Pcomm)}{=} && ((x \odot y)^- \odot x)^- \odot x \\
&\overset{(9)}{=} && (x \odot y) \wedge_m^M x.
\end{aligned}$$

(37) $x \wedge_m^M (x \oplus y)$

$$\overset{(1)}{=} x \wedge_m^M (x^- \odot y^-)^-$$
$$\overset{(9)}{=} (x^- \odot (x^- \odot y^-)^-)^- \odot (x^- \odot y^-)^-$$
$$\overset{(6)}{=} ((x^- \odot (x^- \odot y^-)^-) \oplus (x^- \odot y^-))^-$$
$$\overset{(Scomm),(Pcomm)}{=} ((x^- \odot y^-) \oplus ((x^- \odot y^-)^- \odot x^-))^-$$
$$\overset{(Pom)}{=} (x^-)^- \overset{(DN)}{=} x, \text{ hence } x \leq_m^M (x \oplus y).$$

(38) $x \wedge_m^M y$

$$\overset{(9)}{=} (x^- \odot y)^- \odot y$$
$$\overset{(Pcomm)}{=} y \odot (x^- \odot y) \ \leq_m^M \ y, \text{ by (36)}.$$

(39) $x \vee_m^M y$

$$\overset{(10)}{=} (x \odot y^-) \oplus y$$
$$\overset{(3)}{=} y \oplus (x \odot y^-) \geq^{\wedge_m^M} y, \text{ by (37)}.$$

(40) $x \vee_m^M (y \wedge_m^M x)$

$$\overset{(10)}{=} (x \odot (y \wedge_m^M x)^-) \oplus (y \wedge_m^M x)$$
$$\overset{(16)}{=} (x \odot (y^- \vee_m^M x^-)) \oplus (y \wedge_m^M x)$$
$$\overset{(35)}{=} (x \odot y^-) \oplus (y \wedge_m^M x)$$
$$\overset{(1)}{=} ((x \odot y^-)^- \odot (y \wedge_m^M x)^-)^-$$
$$\overset{(9)}{=} [(x \odot y^-)^- \odot ((y^- \odot x)^- \odot x)^-]^-$$
$$\overset{(Pcomm)}{=} [(x \odot (y^- \odot x)^-)^- \odot (y^- \odot x)^-]^-$$
$$\overset{(9),(DN)}{=} [x^- \wedge_m^M (y^- \odot x)^-]^-$$
$$\overset{(1)}{=} [x^- \wedge_m^M (y \oplus x^-)]^-$$
$$\overset{(Scomm)}{=} [x^- \wedge_m^M (x^- \oplus y)]^- \overset{(37)}{=} (x^-)^- = x.$$

(41) Since $x \ \leq_m^M \ y \Longleftrightarrow x \wedge_m^M y = x$, it follows that

$$y \vee_m^M x = y \vee_m^M (x \wedge_m^M y) \overset{(40)}{=} y.$$

(42) If $x \ \leq_m^M \ y$, then $y \vee_m^M x = y$, by (41); then,

$$y^- = (y \vee_m^M x)^- \overset{(15)}{=} y^- \wedge_m^M x^-, \text{ i.e. } y^- \ \leq_m^M \ x^-.$$

(43) If $x \leq_m^M y$, then $y = y \vee_m^M x$, by (41). Then,

$$\begin{aligned} &(x \oplus z) \wedge_m^M (y \oplus z) \\ &= (x \oplus z) \wedge_m^M ((y \vee_m^M x) \oplus z) \\ &\stackrel{(10)}{=} (x \oplus z) \wedge_m^M (((y \odot x^-) \oplus x) \oplus z) \\ &\stackrel{(Sass)}{=} (x \oplus z) \wedge_m^M ((y \odot x^-) \oplus (x \oplus z)) \\ &\stackrel{(Scomm)}{=} (x \oplus z) \wedge_m^M ((x \oplus z) \oplus (y \odot x^-)) \\ &\stackrel{(37)}{=} x \oplus z. \end{aligned}$$

(44) If $x \leq_m^M y$, then $y^- \leq_m^M x^-$, by (42); it follows, by (43), that:
$(y \odot z)^- \stackrel{(1),(DN)}{=} y^- \oplus z^- \leq_m^M x^- \oplus z^- \stackrel{(1),(DN)}{=} (x \odot z)^-$;
hence, $x \odot z \leq_m^M y \odot z$, by (42) again.

(45) $(x \wedge_m^M y) \wedge_m^M (y \wedge_m^M z)$

$$\begin{aligned} &\stackrel{(9)}{=} [(x \wedge_m^M y)^- \odot (y \wedge_m^M z)]^- \odot (y \wedge_m^M z) \\ &\stackrel{(1)}{=} [(x \wedge_m^M y) \oplus (y \wedge_m^M z)^-] \odot (y \wedge_m^M z) \\ &\stackrel{(16),(9)}{=} [(x \wedge_m^M y) \oplus (y^- \vee_m^M z^-)] \odot ((y^- \odot z)^- \odot z) \\ &\stackrel{(10)}{=} [(x \wedge_m^M y) \oplus ((y^- \odot z) \oplus z^-)] \odot (y^- \odot z)^- \odot z \\ &\stackrel{(1)}{=} [(x \wedge_m^M y) \oplus ((y^- \odot z) \oplus z^-)] \odot (y \oplus z^-) \odot z \\ &\stackrel{(Sass),(1),(Pass)}{=} [[(x \wedge_m^M y) \oplus z^- \oplus (y \oplus z^-)^-] \odot (y \oplus z^-)] \odot z \\ &\stackrel{(1)}{=} ([((x \wedge_m^M y) \oplus z^-)^- \odot (y \oplus z^-)]^- \odot (y \oplus z^-)) \odot z \\ &\stackrel{(9)}{=} (((x \wedge_m^M y) \oplus z^-) \wedge_m^M (y \oplus z^-)) \odot z \\ &= ((x \wedge_m^M y) \oplus z^-) \odot z \\ &\stackrel{(1)}{=} ((x \wedge_m^M y)^- \odot z)^- \odot z \\ &\stackrel{(9)}{=} (x \wedge_m^M y) \wedge_m^M z, \end{aligned}$$

since $x \wedge_m^M y \leq_m^M y$, by (38), implies $(x \wedge_m^M y) \oplus z^- \leq_m^M y \oplus z^-$, by (43), i.e. $((x \wedge_m^M y) \oplus z^-) \wedge_m^M (y \oplus z^-) = (x \wedge_m^M y) \oplus z^-$.

(46) $(x \vee_m^M y) \vee_m^M (y \vee_m^M z)$

$$\begin{aligned} &\stackrel{(10)}{=} [(x \vee_m^M y) \odot (y \vee_m^M z)^-] \oplus (y \vee_m^M z) \\ &\stackrel{(15)}{=} [(x \vee_m^M y) \odot (y^- \wedge_m^M z^-)] \oplus (y \vee_m^M z) \\ &\stackrel{(9),(DN)}{=} [(x \vee_m^M y) \odot ((y \odot z^-)^- \odot z^-)] \oplus (y \vee_m^M z) \\ &\stackrel{(10),(Pcomm),(Pass)}{=} [((x \vee_m^M y) \odot z^-) \odot (y \odot z^-)^-] \oplus ((y \odot z^-) \oplus z) \end{aligned}$$

$$\begin{array}{ll}
\overset{(Sass)}{=} & ([((x \vee_m^M y) \odot z^-) \odot (y \odot z^-)^-] \oplus (y \odot z^-)) \oplus z \\
\overset{(10)}{=} & [((x \vee_m^M y) \odot z^-) \vee_m^M (y \odot z^-)] \oplus z \\
= & [(x \vee_m^M y) \odot z^-] \oplus z \\
\overset{(10)}{=} & (x \vee_m^M y) \vee_m^M z,
\end{array}$$

since $y \leq_m^M x \vee_m^M y$, by (39), implies $y \odot z^- \leq_m^M (x \vee_m^M y) \odot z^-$, by (44), and hence $((x \vee_m^M y) \odot z^-) \vee_m^M (y \odot z^-) = (x \vee_m^M y) \odot z^-$, by (41). □

Consider the following properties (see [27], Proposition 3.13):

(Pq) $x \odot [y \vee_m^M (z \vee_m^M x^-)] = (x \odot y) \vee_m^M (x \odot z)$ and

(Pqq) $x \odot [y \vee_m^M (x \odot z)^-] = (x \odot y) \vee_m^M (x \odot (x \odot z)^-)$.

Lemma 3.2 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pq) \Longrightarrow (35) \text{ and } (Pqq) \Longrightarrow (35).$$

Proof: Take $z = 0$ in (Pq) to obtain (35).
Take $z = 1$ in (Pqq) to obtain (35). □

Theorem 3.3 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pq) \iff (Pqq).$$

Proof: First, we prove:

$$z \vee_m^M x^- = (x \odot (x \odot z)^-)^-. \qquad \text{(a)}$$

Indeed, $z \vee_m^M x^- = (z \odot x^=) \oplus x^- \overset{(DN)}{=} (z \odot x) \oplus x^- = ((z \odot x)^- \odot x^=)^-$ $= ((z \odot x)^- \odot x)^- \overset{(Pcomm)}{=} (x \odot (x \odot z)^-)^-$.

$(Pq) \Longrightarrow (Pqq)$ $(x \odot y) \vee_m^M (x \odot (x \odot z)^-)$

$$\begin{array}{ll}
\overset{(Pq)}{=} & x \odot [y \vee_m^M (Z \vee_m^M x^-)], \text{ where } Z := (x \odot z)^-, \\
\overset{(a)}{=} & x \odot [y \vee_m^M (x \odot (x \odot Z)^-)^-] \\
= & x \odot [y \vee_m^M (x \odot (x \odot (x \odot z)^-)^-)^-] \\
\overset{(a)}{=} & x \odot [y \vee_m^M (x \odot (z \vee_m^M x^-))^-] \\
\overset{(35)}{=} & x \odot [y \vee_m^M (x \odot z)^-].
\end{array}$$

$$
\begin{aligned}
(Pqq) \implies (Pq)\ & x \odot [y \vee_m^M (z \vee_m^M x^-)] \\
& \stackrel{(a)}{=} x \odot [y \vee_m^M (x \odot (x \odot z)^-)^-] \\
& \stackrel{(Pqq)}{=} (x \odot y) \vee_m^M (x \odot (x \odot Z)^-), \text{ where } Z := (x \odot z)^-, \\
& \stackrel{(a)}{=} (x \odot y) \vee_m^M (x \odot (x \odot (x \odot z)^-)^-) \\
& \stackrel{(a)}{=} (x \odot y) \vee_m^M (x \odot (z \vee_m^M x^-)) \\
& \stackrel{(35)}{=} (x \odot y) \vee_m^M (x \odot z).
\end{aligned}
$$

□

By Theorem 3.3 and since (Pom) $\iff$ (Pq) ([27], Theorem 3.26), we obtain:

Corollary 3.4 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive left-m-BE algebra. Then,*

$$(Pom) \iff (Pq) \iff (Pqq).$$

Proposition 3.5 *(See [27], Proposition 3.18 for QMV algebras)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. We have:

$$x \vee_m^M y \ \leq_m^M x \oplus y, \tag{47}$$

$$x \odot y \ \leq_m^M x \wedge_m^M y. \tag{48}$$

Proof:

(47) Since $x \odot y^- \ \leq_m^M \ x$, by (36), then $x \vee_m^M y = (x \odot y^-) \oplus y \ \leq_m^M \ x \oplus y$, by (43).

(48) Since $x^- \odot y \ \leq_m^M \ x^-$, by (36), then $x \ \leq_m^M \ (x^- \odot y)^-$, by (42) and (DN); hence, $x \odot y \ \leq_m^M \ (x^- \odot y)^- \odot y = x \wedge_m^M y$, by (44).

□

Proposition 3.6 *(See [27], Proposition 3.19 for QMV algebras)*
Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. We have:

$$x \wedge_m^M ((x \oplus y) \wedge_m^M z) = x \wedge_m^M z \quad (\textit{absorption law } 1), \tag{49}$$

$$x \vee_m^M ((x \odot y) \vee_m^M z) = x \vee_m^M z \quad (\textit{absorption law } 2), \tag{50}$$

$$x \leq_m^M z^-,\ y \leq_m^M z^-,\ x \oplus z = y \oplus z \Longrightarrow x = y \quad (\textit{cancellation law } 1), \tag{51}$$

$$z^- \leq_m^M x,\ z^- \leq_m^M y,\ x \odot z = y \odot z \Longrightarrow x = y \quad (\textit{cancellation law } 2), \tag{52}$$

$$x \ \leq_m^M \ y \Longrightarrow x \wedge_m^M z \ \leq_m^M \ y \wedge_m^M z \quad (\textit{monotonicity of } \wedge_m^M), \tag{53}$$

$$x \ \leq_m^M \ y \Longrightarrow x \vee_m^M z \ \leq_m^M \ y \vee_m^M z \quad (\textit{monotonicity of } \vee_m^M), \tag{54}$$

$$x \ \leq_m^M \ y,\ y \ \leq_m^M \ z \Longrightarrow x \ \leq_m^M \ z \quad (\textit{transitivity of } \leq_m^M). \tag{55}$$

Proof: The same as for QMV algebras, namely:

(49) $x \wedge_m^M ((x \oplus y) \wedge_m^M z)$

$$\overset{(9)}{=} (x^- \odot ((x \oplus y) \wedge_m^M z))^- \odot ((x \oplus y) \wedge_m^M z)$$
$$\overset{(1),(9)}{=} (x \oplus ((x \oplus y) \wedge_m^M z)^-) \odot (((x \oplus y)^- \odot z)^- \odot z)$$
$$\overset{(16),(1)}{=} (x \oplus ((x \oplus y)^- \vee_m^M z^-)) \odot (((x \oplus y) \oplus z^-) \odot z)$$
$$\overset{(10)}{=} (x \oplus (((x \oplus y)^- \odot z) \oplus z^-)) \odot ((x \oplus y \oplus z^-) \odot z)$$
$$\overset{(Sass)}{=} ((x \oplus z^-) \oplus (x \oplus y \oplus z^-)^-) \odot ((x \oplus y \oplus z^-) \odot z)$$
$$\overset{(1)}{=} ((x \oplus z^-)^- \odot (x \oplus y \oplus z^-))^- \odot (x \oplus y \oplus z^-) \odot z$$
$$\overset{(Pass)}{=} [((x \oplus z^-)^- \odot (x \oplus y \oplus z^-))^- \odot (x \oplus y \oplus z^-)] \odot z$$
$$\overset{(9)}{=} ((x \oplus z^-) \wedge_m^M (x \oplus y \oplus z^-)) \odot z$$
$$= (x \oplus z^-) \odot z$$
$$\overset{(1)}{=} (x^- \odot z)^- \odot z$$
$$\overset{(9)}{=} x \wedge_m^M z,$$

since $x \oplus z^- \leq_m^M x \oplus z^- \oplus y$, by (37), implies $(x \oplus z^-) \wedge_m^M (x \oplus z^- \oplus y) = x \oplus z^-$.

(50) $x \vee_m^M ((x \odot y) \vee_m^M z)$

$$\overset{(10)}{=} (x \odot ((x \odot y) \vee_m^M z)^-) \oplus ((x \odot y) \vee_m^M z)$$
$$\overset{(15),(10)}{=} (x \odot ((x \odot y)^- \wedge_m^M z^-)) \oplus (((x \odot y) \odot z^-) \oplus z)$$
$$\overset{(9)}{=} (x \odot (((x \odot y) \odot z^-)^- \odot z^-)) \oplus ((x \odot y \odot z^-) \oplus z)$$
$$\overset{(Pass)}{=} ((x \odot z^-) \odot (x \odot y \odot z^-)^-) \oplus ((x \odot y \odot z^-) \oplus z)$$
$$\overset{(Sass)}{=} [((x \odot z^-) \odot (x \odot y \odot z^-)^-) \oplus (x \odot y \odot z^-)] \oplus z$$
$$\overset{(1)}{=} [(x \odot z^-) \vee_m^M (x \odot y \odot z^-)] \oplus z$$
$$\overset{(41)}{=} (x \odot z^-) \oplus z$$
$$\overset{(10)}{=} x \vee_m^M z,$$

since $x \odot y \odot z^- \leq_m^M x \odot z^-$, by (36), implies $(x \odot z^-) \vee_m^M (x \odot y \odot z^-) = x \odot z^-$, by (41).

(51) $x \leq_m^M z^-$ and $y \leq_m^M z^-$ mean $x \wedge_m^M z^- = x$ and $y \wedge_m^M z^- = y$. Then, $x = x \wedge_m^M z^- \overset{(9)}{=} (x^- \odot z^-)^- \odot z^- \overset{(1)}{=} (x \oplus z) \odot z^- = (y \oplus z) \odot z^-$

$\overset{(1)}{=} (y^- \odot z^-)^- \odot z^- \overset{(9)}{=} y \wedge_m^M z^- = y$.

(52) $z^- \leq_m^M x$ and $z^- \leq_m^M y$ imply $x \vee_m^M z^- = x$ and $y \vee_m^M z^- = y$, by (41). Then, $x = x \vee_m^M z^- \overset{(10),(DN)}{=} (x \odot z) \oplus z^- = (y \odot z) \oplus z^- \overset{(10)}{=} y \vee_m^M z^- = y$.

(53) $x \leq_m^M y$ implies $x \oplus z^- \leq_m^M y \oplus z^-$, by (43), and hence $(x \oplus z^-) \odot z \leq_m^M (y \oplus z^-) \odot z$, by (44). Then, $x \wedge_m^M z \overset{(9)}{=} (x^- \odot z)^- \odot z = (x \oplus z^-) \odot z \leq_m^M (y \oplus z^-) \odot z \overset{(1)}{=} (y^- \odot z)^- \odot z = y \wedge_m^M z$.

(54) $x \leq_m^M y$ implies $x \odot z^- \leq_m^M y \odot z^-$, by (44), hence $(x \odot z^-) \oplus z \leq_m^M (y \odot z^-) \oplus z$, by (43). Then, $x \vee_m^M z \overset{(10)}{=} (x \odot z^-) \oplus z \leq_m^M (y \odot z^-) \oplus z \overset{(10)}{=} y \vee_m^M z$.

(55) $x \leq_m^M y$ and $y \leq_m^M z$ mean $x = x \wedge_m^M y$ and $y = y \wedge_m^M z$. Then, $x = x \wedge_m^M y = (x \wedge_m^M y) \wedge_m^M (y \wedge_m^M z) \overset{(45)}{=} (x \wedge_m^M y) \wedge_m^M z = x \wedge_m^M z$; thus, $x \leq_m^M z$. □

Corollary 3.7 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-OM algebra. The binary relation $\leq_m^M$ is an order relation.*

Proof: The reflexivity and the antisymmetry follow by Corollary 2.10, while the transitivity follows by (55). □

Proposition 3.8 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-QMV algebra. Then, $\mathcal{A}^L$ is a left-OM algebra.*

Proof: By Theorem 2.13 (1). □

Proposition 3.9 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a left-MV algebra. Then, $\mathcal{A}^L$ is a left-OM algebra, i.e.* **MV** $\subset$ **OM**.

Proof: By ([27], Theorem 4.20). □

Remark 3.10 In a left-OM algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:

- the initial binary relation, $\leq_m$ ($x \leq_m y \iff x \odot y^- = 0$), is only reflexive ((m-Re) holds);
- the binary relation $\leq_m^M$ ($x \leq_m^M y \iff x \wedge_m^M y = x$) is an order, by Corollary 3.7, but not a lattice order with respect to $\wedge_m^M$, $\vee_m^M$, since $x \wedge_m^M y \neq y \wedge_m^M x$;
- the binary relation $\leq_m^P$ ($x \leq_m^P y \iff x \odot y = x$) is only antisymmetric and transitive, by ([25], Proposition 3.11).

3.2 Puting Orthomodular Algebras on the "map"

We have the connections from the Figure 3.

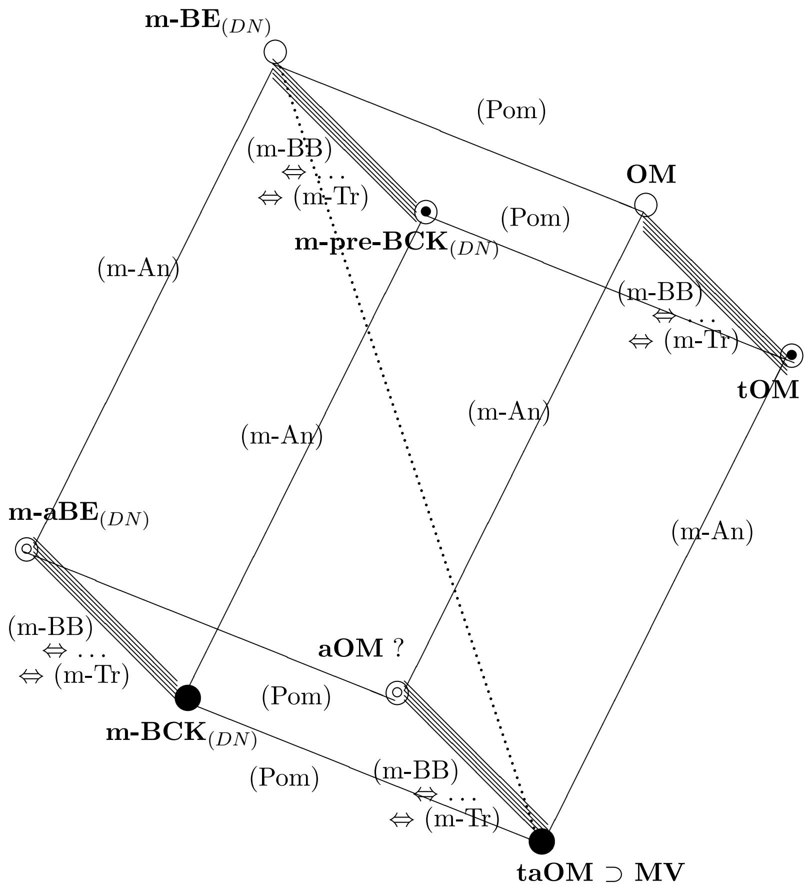


Figure 3: Putting **OM**, **tOM**, **aOM** and **taOM** on the "map", where ? means that there is an open problem concerning **aOM**

4 A New Algebra: The Trans Algebra (TRANS)

Concerning the involutive m-BE algebras not verifying (m-An), note that the examples: ([27], Example 6.2) of PreMV algebra, ([27], Example 6.3) of MMV algebra, ([27], Example 6.6) of tPreMV algebra and ([27], Example 6.7) of tMMV algebra have the binary relation $\leq_m^M$ transitive, hence an order relation (that is not a lattice order since $x \wedge_m^M y \neq y \wedge_m^M x$ for some x, y). But this is not true in general. We present, in Section 6, Example 6.3 of PreMV algebra, Example 6.4 of MMV algebra, Example 6.7 of tPreMV algebra and Example 6.8 of tMMV algebra whose binary relation $\leq_m^M$ is not transitive. We present, in Section 6 also, Example 6.2 of involutive m-BE algebra and Example 6.6 of involutive m-pre-BCK algebra having the binary relation $\leq_m^M$ transitive, hence an order relation (that is not a lattice order since $x \wedge_m^M y \neq y \wedge_m^M x$ for some x, y), but also Example 6.1 of involutive m-BE algebra and Example 6.5 of involutive m-pre-BCK algebra having $\leq_m^M$ not transitive.

Concerning the involutive m-aBE algebras, we have a surprising result, the next Theorem 4.3 (saying that any m-BCK algebra has the binary relation $\leq_m^M$ transitive, hence an order relation), **obtained by** *Prover9* [29] **in about 4 seconds and in 29 steps (the length of proof is 29); we have grouped the steps of proof in the following Lemmas 4.1, 4.2 and Theorem 4.3.**

Lemma 4.1 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra. Then,*

$$x \odot (y \odot (y \odot x)^-) = 0, \tag{56}$$

$$(x \odot y)^- \odot (x \odot (x \odot (x \odot y)^-)^-) = 0. \tag{57}$$

Proof:

(56) By (m-Re), $(x \odot y) \odot (x \odot y)^- = 0$, hence by (Pass) and (Pcomm), we obtain (56).

(57) In (56), take $X := (x \odot y)^-$ and $Y := z$ to obtain:

$$(x \odot y)^- \odot (z \odot (z \odot (x \odot y)^-)^-) = 0; \tag{a}$$

take then $z := x$ in (a) to obtain (57). □

Lemma 4.2 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an involutive m-BE algebra verifying (m-BB) (i.e. an involutive m-tBE algebra (= m-pre-BCK algebra)). Then,*

$$(x \odot y)^- \odot (z \odot (y \odot (z \odot x^-)^-)) = 0, \tag{58}$$

$$(x \odot y)^- \odot (z \odot (x \odot (z \odot y^-)^-)) = 0, \tag{59}$$

$$x \odot (y \odot (x \odot (z \odot (z \odot y)^-)^-)^-) = 0, \tag{60}$$

$$x \odot (y \odot (x \odot (x \odot (x \odot y)^-)^-)^-) = 0. \tag{61}$$

Proof:

(58) From (m-BB) $((x \odot y)^- \odot (z \odot y)) \odot (z \odot x^-)^- = 0)$, by (Pass) we obtain (58).

(59) In (58), interchange x with y and apply (Pcomm) to obtain (59).

(60) In (59), take $X := y$, $Y := z \odot (z \odot y)^-$ and $Z := x$ to obtain:

$$(y \odot (z \odot (z \odot y)^-))^- \odot (x \odot (y \odot (x \odot (z \odot (z \odot y)^-)^-)^-)) = 0; \quad \text{(a)}$$

note that, in (a), $y \odot (z \odot (z \odot y)^-) \overset{(56)}{=} 0$, hence, by (Neg0-1), (PU), (a) becomes (60).

(61) In (60), take $z := x$ to obtain (61). □

Theorem 4.3 *Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a (involutive) m-BCK algebra. Then, the binary relation $\leq_m^M$ is transitive.*

Proof: Suppose $C1 \leq_m^M C2$ and $C2 \leq_m^M C3$, i.e. $C1 \wedge_m^M C2 = C1$ and $C2 \wedge_m^M C3 = C2$, i.e.:

$$C2 \odot (C2 \odot C1^-)^- = C1 \quad \textit{and} \tag{62}$$

$$C3 \odot (C3 \odot C2^-)^- = C2. \tag{63}$$

We must prove that $C1 \leq_m^M C3$, i.e. $C1 \wedge_m^M C3 = C1$, i.e.:

$$C3 \odot (C3 \odot C1^-)^- = C1. \tag{64}$$

In (m-An) ($y \odot x^- = 0$ and $x \odot y^- = 0$ imply $x = y$), take $X := x \odot y$ and $Y := z$ to obtain:

$$(x \odot y)^- \odot z = 0 \textit{ and } x \odot (y \odot z^-) = 0 \textit{ imply } x \odot y = z. \tag{65}$$

In (65), take $Z := x \odot (x \odot (x \odot y)^-)^-$ to obtain:

$$(x \odot y)^- \odot [x \odot (x \odot (x \odot y)^-)^-] = 0 \textit{ and} \tag{66}$$

$$x \odot (y \odot [x \odot (x \odot (x \odot y)^-)^-]^-) = 0 \textit{ imply} \tag{67}$$

$$x \odot y = x \odot (x \odot (x \odot y)^-)^-. \tag{68}$$

Note that (66) is true by (57) and (67) is true by (61). It follows that (68) holds. It follows that:

$$x \odot (x \odot (x \odot y)^-)^- = x \odot y. \tag{69}$$

Now, from (63), by multiplying by x on the right side and by (Pcomm), we obtain:

$$C3 \odot ((C3 \odot C2^-)^- \odot x) = C2 \odot x. \tag{70}$$

Next, in (69), take $X := C3$ and $Y := (C3 \odot C2^-)^- \odot x$ to obtain:

$$C3 \odot (C3 \odot (C3 \odot ((C3 \odot C2^-)^- \odot x))^-)^- = C3 \odot ((C3 \odot C2^-)^- \odot x). \tag{71}$$

Note that, in (71), we have twice that $C3 \odot ((C3 \odot C2^-)^- \odot x) \overset{(70)}{=} C2 \odot x$, hence (71) becomes:

$$C3 \odot (C3 \odot (C2 \odot x)^-)^- = C2 \odot x. \tag{72}$$

Next, in (72), take $X := (C2 \odot C1^-)^-$ to obtain:

$$C3 \odot (C3 \odot (C2 \odot (C2 \odot C1^-)^-)^-)^- = C2 \odot (C2 \odot C1^-)^-. \tag{73}$$

Note that, in (73), we have twice that $C2 \odot (C2 \odot C1^-)^- \overset{(62)}{=} C1$, hence (73) becomes (64). □

We present, in Section 6, Example 6.12 of involutive m-aBE algebra having $\leq_m^M$ transitive (hence an order relation, by Corollary 2.10), but also Example 6.11 of involutive m-aBE algebra having $\leq_m^M$ not transitive.

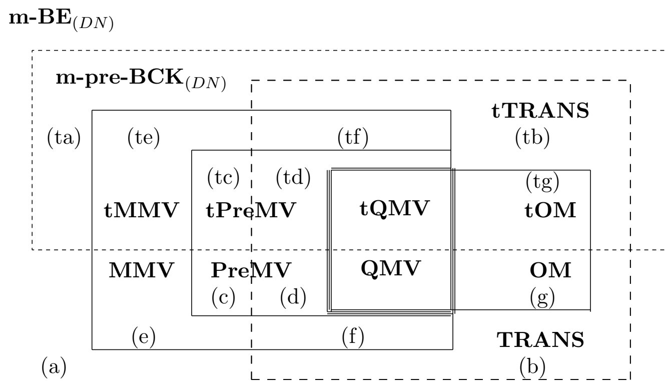
Since the property of having $\leq_m^M$ transitive (hence an order relation) is so spread, we shall introduce the following new algebra:

Definition 4.4 (The dual one is omitted)

A *left-trans algebra* is an involutive left-m-BE algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ having the binary relation $\leq_m^M$ transitive, i.e. verifying the property:

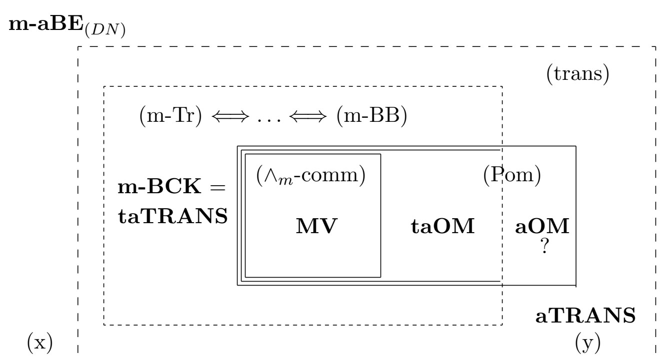
(trans) $x \leq_m^M y$ and $y \leq_m^M z$ imply $x \leq_m^M z$, for all $x, y, z \in A^L$.

It follows that trans algebras are those involutive m-BE algebras having $\leq_m^M$ an order relation, by Corollary 2.10. We shall denote by **TRANS** the class of all left-trans algebras. Note that **taTRANS** = **m-BCK**, by above Theorem 4.3. Hence, we have the connections from the Figures 4 and 5 (see Figures 1 and 2, respectively).



where:
(a): Example 6.1 (e): Example 6.4 (ta): Example 6.5 (te): Example 6.8
(b): Example 6.2 (f): [27], Example 6.3 (tb): Example 6.6 (tf): [27], Example 6.7
(c): Example 6.3 (g): Example 6.9 (tc): Example 6.7 (tg): Example 6.10
(d): [27], Example 6.2 (td): [27], Example 6.6

Figure 4: Resuming connections between **OM**, **PreMV**, **MMV**, **QMV** and **m-pre-BCK**$_{(DN)}$, **TRANS**



where:
(x): Example 6.11 (y): Example 6.12

Figure 5: Resuming connections between **MV**, **taOM**, **aOM**, **m-BCK** and **aTRANS**, where ? means that there is an open problem concerning **aOM**

Remark 4.5 In a left-trans algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:

- the initial binary relation, $\leq_m$ ($x \leq_m y \iff x \odot y^- = 0$), is only reflexive ((m-Re) holds);

- the binary relation $\leq_m^M$ $(x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x)$ is an order, by definition and Corollary 2.10, but not a lattice order with respect to $\wedge_m^M$, $\vee_m^M$, since $x \wedge_m^M y \neq y \wedge_m^M x$;
- the binary relation $\leq_m^P$ $(x \leq_m^P y \Longleftrightarrow x \odot y = x)$ is only antisymmetric and transitive, by ([25], Proposition 3.11).

5 The taOM Algebras Inside the m-BCK Algebras

Note that a *taOM algebra* is a transitive antisymmetric involutive m-BE algebra verifying (Pom), hence it is a (involutive) m-BCK algebra verifying (Pom), so we could say that it is an *orthomodular m-BCK algebra*.

First, we shall analyse more deeply the (involutive) m-BCK algebras.

5.1 The m-BCK Algebras

We have seen, in the previous section, that any m-BCK algebra has $\leq_m^M$ an order relation, by Theorem 4.3 and Corollary 2.10. This order relation $\leq_m^M$ is a lattice order if and only if the property ($\wedge_m$-comm) holds, i.e. if and only if the m-BCK algebra is an MV algebra, and in MV algebras, $x \leq_m^M y \Longleftrightarrow x \leq_m y$ and both $\leq_m^M$ and $\leq_m$ are distributive lattice orders.

But any m-BCK algebra has $\leq_m$ as an order relation too, since (m-Re), (m-An) and (m-Tr) hold. But **we do not know when (what property determines that) this order relation $\leq_m$ $(\Longleftrightarrow \leq_m^B)$ is a lattice order**; we have examples of m-BCK algebras that are not lattices and examples that are lattices [1, 31], distributive or not.

We shall denote by **m-BCK-L** the class of all left-m-BCK lattices.
We have: **m-BCK-L** $\subset$ **m-BCK**.

Recall the following definition ([17], Definition 1.2.9):

Definition 5.1 (The dual one is omitted)

An *involutive residuated left-lattice*, or a *left-IRL* for short, is a bounded residuated left-lattice satisfying (DN), i.e. is an algebra

$\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:

(irl1) $(A^L, \wedge, \vee, 0, 1)$ is a bounded lattice w.r. to the lattice order $\leq$,

(irl2) $(A^L, \odot, 1)$ is an abelian monoid (i.e. (PU), (Pcomm), (Pass) hold), (RP) for all $x, y, z \in A^L$, $\quad x \leq y \to z \Longleftrightarrow x \odot y \leq z$,

(DN) for all $x \in A^L$, $\quad (x^-)^- = x$ (or $x^= = x$), where $x^- \stackrel{def.}{=} x \to 0$.

Note that (irl1) means that:

- $\leq$ is an order (i.e. it is reflexive (Re), antisymmetric (An) and transitive (Tr)) and
- for all $x, y \in A^L$, $\exists\ x \wedge y = \inf(x, y)$ and $\exists\ x \vee y = \sup(x, y)$ and
- for all $x \in A^L$, $0 \leq x \leq 1$, i.e. we have:

  (F) (first element) $0 \leq x$ and

  (L) (last element) $x \leq 1$.

We shall denote by **IRL** the class of all involutive residuated left-lattices. We shall prove that involutive residuated left-lattices (**IRL**) are definitionally equivalent (d. e.) to left-m-BCK-lattices (**m-BCK-L**). First, we prove some properties of left-IRLs.

Proposition 5.2 *Let $\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \to, 0, 1)$ be a left-IRL. We have: for all $x, y \in A^L$,*

(V) $\quad x \leq 0 \Longleftrightarrow x = 0$,
(N) $\quad 1 \leq x \Longleftrightarrow x = 1$,
(EqrelR) $\quad x \leq y \Longleftrightarrow x \to y = 1$,
(irl3) $\quad x \leq y \Longleftrightarrow x \leq_m y$, *where* $x \leq_m y \stackrel{def.}{\Longleftrightarrow} x \odot y^- = 0$, *hence* *(Re)* $\Longleftrightarrow$ *(m-Re)*, *(An)* $\Longleftrightarrow$ *(m-An)*, *(Tr)* $\Longleftrightarrow$ *(m-Tr)*, *(L)* $\Longleftrightarrow$ *(m-L) and* $x \wedge y = x \wedge_m y = \inf_m(x, y)$, $x \vee y = x \vee_m y = \sup_m(x, y)$,
(Neg0-1) $\quad 0^- = 1$,
(Neg1-0) $\quad 1^- = 0$,
(m-L) $\quad x \odot 0 = 0$,
(m-Re) $\quad x \odot x^- = 0$,
(irl4) $\quad x \odot (x \to y) \leq y$,
(irl5) $\quad x \to y = (x \odot y^-)^-$,
(irl6) $\quad x \odot y = (x \to y^-)^-$,
(irl7) $\quad$ *the algebra* $(A^L, \odot, ^-, 1)$ *is a (involutive) left-m-BCK lattice, with the lattice order* $\leq_m \Longleftrightarrow \leq$ *and the lattice operations* $\wedge_m = \wedge$, $\vee_m = \vee$.

Proof:

(V) Since $0 \leq x$, by (F), then, if $x \leq 0$, we obtain, by (An), $x = 0$. Conversely, if $x = 0$, then $x \leq 0$, by (Re).

(N) Since $x \leq 1$, by (L), then, if $1 \leq x$, we obtain, by (An), $x = 1$. Conversely, if $x = 1$, then $1 \leq x$, by (Re).

(EqrelR) $x \to y = 1 \overset{(N)}{\Longleftrightarrow} 1 \leq x \to y \overset{(RP)}{\Longleftrightarrow} 1 \odot x \leq y \overset{(PU)}{\Longleftrightarrow} x \leq y$.

(irl3) $x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0 \overset{(V)}{\Longleftrightarrow} x \odot y^- \leq 0 \overset{(RP)}{\Longleftrightarrow} x \leq y^- \to 0 = y^= \overset{(DN)}{=} y$; the rest follows by this equivalence.

(Neg0-1) $0^- \overset{def.}{=} 0 \to 0 = 1 \overset{(EqrelR)}{\Longleftrightarrow} 0 \leq 0$, which is true by (Re).

(Neg1-0) $1^- \overset{(Neg0-1)}{=} (0^-)^- \overset{(DN)}{=} 0$.

(m-L) (direct proof) $x \odot 0 = 0 \overset{(V)}{\Longleftrightarrow} x \odot 0 \leq 0 \overset{(RP)}{\Longleftrightarrow} x \leq 0 \to 0 = 0^- \overset{(Neg0-1)}{=} 1$, that is true by (L).

(m-Re) (direct proof) $x \odot x^- = 0 \overset{(V)}{\Longleftrightarrow} x \odot x^- \leq 0 \overset{(RP)}{\Longleftrightarrow} x \leq x^- \to 0 = x^= \overset{(DN)}{=} x$, that is true by (Re).

(irl4) $x \odot (x \to y) \leq y \overset{(Pcomm)}{\Longleftrightarrow} (x \to y) \odot x \leq y \overset{(RP)}{\Longleftrightarrow} x \to y \leq x \to y$, that is true by (Re).

(irl5) First, we prove

$$x \to y \leq (x \odot y^-)^-. \tag{a}$$

Indeed, $x \to y \leq (x \odot y^-)^- \overset{(irl3)}{\Longleftrightarrow} (x \to y) \odot (x \odot y^-)^= = 0 \overset{(DN)}{\Longleftrightarrow} (x \to y) \odot (x \odot y^-) = 0 \overset{(Pass),(Pcomm)}{\Longleftrightarrow} (x \odot (x \to y)) \odot y^- = 0 \overset{(irl3)}{\Longleftrightarrow} x \odot (x \to y) \leq y$, that is true by (irl4); thus, (a) holds.

Then, we prove

$$(x \odot y^-)^- \leq x \to y. \tag{b}$$

Indeed, $(x \odot y^-)^- \leq x \to y \overset{(RP)}{\Longleftrightarrow} (x \odot y^-)^- \odot x \leq y \overset{(irl3)}{\Longleftrightarrow}$ $((x \odot y^-)^- \odot x) \odot y^- = 0 \overset{(Pass)}{\Longleftrightarrow} (x \odot y^-)^- \odot (x \odot y^-) = 0$, that is true by (Pcomm) and (m-Re); thus, (b) holds.

By (a), (b) and (An), we obtain (irl5).

(irl6) $(x \to y^-)^- \overset{(irl5)}{=} (x \odot y^=)^= \overset{(DN)}{=} x \odot y$.

(irl7) By (Neg1-0), (irl2), (m-L), (DN), (m-Re), (m-An), (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), $(A^L, \odot, ^-, 1)$ is a (involutive) left-m-BCK algebra; since $\leq$ is a lattice order, by (irl1), it follows that $\leq_m$ is a lattice order, by (irl3), hence, $(A^L, \odot, ^-, 1)$ is a (involutive) left-m-BCK lattice, with the lattice operations $\wedge_m = \wedge$ and $\vee_m = \vee$. □

Now, we prove the definitional equivalence (d. e.) between **IRL** and **m-BCK-L**.

Theorem 5.3

1) Let $\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \to, 0, 1)$ be an involutive residuated left-lattice.

Define $f(\mathcal{A}^L) = (A^L, \odot, ^-, 1)$, where $x^- \overset{def.}{=} x \to 0$.

Then, $f(\mathcal{A}^L)$ is a (involutive) left-m-BCK lattice, with the lattice order $\leq_m \Longleftrightarrow \leq$ and the lattice operations $\wedge_m = \wedge$, $\vee_m = \vee$.

(1') Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be a (involutive) left-m-BCK lattice, with the lattice order $\leq_m$ ($x \leq_m y \overset{def.}{\Longleftrightarrow} x \odot y^- = 0$) and the lattice operations $x \wedge_m y = \inf_m(x, y)$ and $x \vee_m y = \sup_m(x, y)$.

Define $g(\mathcal{A}^L) = (A^L, \wedge_m, \vee_m, \odot, \to, 0, 1)$, where $x \to y \overset{def.}{=} (x \odot y^-)^-$ and $0 \overset{def.}{=} 1^-$.

Then, $g(\mathcal{A}^L)$ is an involutive residuated left-lattice.

(2) The maps f and g are mutually inverse.

Proof:

(1) By Proposition 5.2, (irl7).

(1') Let $\mathcal{A}^L = (A^L, \odot, ^-, 1)$ be an (involutive) left-m-BCK lattice w.r. to the lattice order $\leq_m$ and the lattice operations $\wedge_m$ and $\vee_m$. Then, (irl1), (irl2) and (DN) hold.

Note that $x \to 0 \overset{def.}{=} (x \odot 0^-)^- \overset{(Neg0-1)}{=} (x \odot 1)^- \overset{(PU)}{=} x^-$.

We prove that (RP) holds. Indeed, first, by ([22], (mCDN2)), (DN) + (Pcomm) $\Longrightarrow$ (DN4), where: (DN4) $x \leq_m y \Longleftrightarrow y^- \leq_m x^-$;

then, $x \leq_m y \to z \Longleftrightarrow x \leq_m (y \odot z^-)^- \stackrel{(DN4)}{\Longleftrightarrow} (y \odot z^-)^= \leq_m x^- \stackrel{(DN)}{\Longleftrightarrow}$ $y \odot z^- \leq_m x^- \Longleftrightarrow (y \odot z^-) \odot x^= = 0 \stackrel{(DN),(Pcomm),(Pass)}{\Longleftrightarrow} (x \odot y) \odot z^- =$ $0 \Longleftrightarrow x \odot y \leq_m z$; thus, (RP) holds.

Hence, $(A^L, \wedge_m, \vee_m, \odot, \to, 0, 1)$ is an involutive residuated left-lattice.

(2) Routine, by Proposition 5.2. □

We write: **IRL $\cong$ m-BCK-L**.

Remark 5.4 In a left-m-BCK algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:

- the initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is an order relation (since (m-Re), (m-An), (m-Tr) hold); it can be a lattice order, but we do not know when, in general, excepting the case of MV algebras;
- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an order, by Corollary 2.10 and Theorem 4.3, but not a lattice order, in general, with respect to $\wedge_m^M$, $\vee_m^M$, since $x \wedge_m^M y \neq y \wedge_m^M x$; it is a distributive lattice order if and only if ($\wedge_m$-comm) holds (i.e. $x \wedge_m^M y = y \wedge_m^M x$), i.e. in the case of MV algebras, when $\leq_m \Longleftrightarrow \leq_m^M$;
- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is only antisymmetric and transitive, by ([25], Proposition 3.11); in Boolean algebras, it is a distributive lattice order and $\leq_m^P \Longleftrightarrow \leq_m \Longleftrightarrow \leq_m^M$.

5.2 The IMTL Algebras and the NM Algebras

The IMTL (Involutive Monoidal t-norm based Logic) algebras and the NM (Nilpotent Minimum) algebras were introduced in [6], see also [7], [30].

Definitions 5.5 (See ([17], Definition 1.2.32)) (Definitions 1)
(The dual ones are omitted)

- A *left-IMTL algebra* is a left-IRL $\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \to, 0, 1)$ verifying:

  (prel) (pre-linearity) $(x \to y) \vee (y \to x) = 1$, for all $x, y \in A^L$.
- A *left-NM algebra* is a left-IMTL algebra $\mathcal{A}^L = (A^L, \wedge, \vee, \odot, \to, 0, 1)$ verifying:

  (WNM) (Weak Nilpotent Minimum) $(x \odot y)^- \vee ((x \wedge y) \to (x \odot y)) = 1$, for all $x, y \in A^L$.

We shall denote by **IMTL** the class of all left-IMTL algebras and by **NM** the class of all left-NM algebras. Hence, we have:

IMTL = **IRL** + (prel) $\cong$ **m-BCK-L** + (prel),

NM = **IMTL** + (WNM).

Recall [17] that: **MV** = **IMTL** + (div), where:

(div) (divisibility) $x \wedge y = x \odot (x \to y) = x \wedge_m^B y$, for all x, y.

Recall also [17] that: $_{(WNM)}$**MV** $\stackrel{def.}{=}$ **MV** + (WNM) = **NM** + (div).

By the above d.e. **IRL** $\cong$ **m-BCK-L**, we obtain a second, equivalent definition of IMTL and NM algebras:

Definitions 5.6 (Definitions 2) (The dual ones are omitted)

- A *left-IMTL algebra* is a (involutive) left-m-BCK lattice $\mathcal{A}^L = (A^L, \odot, ^-, 1)$, with the lattice order $\leq_m$ and the lattice operations $\wedge_m = \inf_m$, $\vee_m = \sup_m$, verifying:

  (prel) $(x \to y) \vee_m (y \to x) = 1$, for all $x, y \in A^L$,

  where $x \to y \stackrel{def.}{=} (x \odot y^-)^-$.

- A *left-NM algebra* is a left-IMTL algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$, with the lattice order $\leq_m$ and the lattice operations $\wedge_m$ and $\vee_m$, verifying:

  (WNM) $(x \odot y)^- \vee_m ((x \wedge_m y) \to (x \odot y)) = 1$, for all $x, y \in A^L$,

  where $x \to y \stackrel{def.}{=} (x \odot y^-)^-$.

Hence, we have the connections from the Figure 6.

Remark 5.7 There is a connection between above (prel) and (prel$_m$) from [27], where:

(prel$_m$) $(x \to y) \vee_m^B (y \to x) = 1$

and between above (WNM) and (aWNM$_m$) from [26], where:

(aWNM$_m$) $(x \odot y) \odot [x \odot (x \odot y^-)^- \odot (x \odot y)^-]^- = x \odot y$,

namely: in MV algebras, they coincide, (prel) = (prel$_m$) and (WNM) = (aWNM$_m$).

5.3 The taOM Algebras

Note that all the examples of finite taOM algebras we found are lattices w.r. to $\leq_m$. The examples of proper taOM lattices (i.e. not being MV algebras or $_{(WNM)}$MV algebras) we found are either NM algebras, or proper IMTL

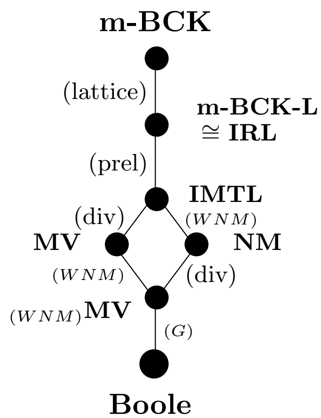


Figure 6: The connections between **m-BCK**, **m-BCK-L**, **IMTL**, **MV**, **NM** and **Boole**

algebras (i.e. not being NM algebras), or proper m-BCK lattices (i.e. not IMTL algebras) - distributive or not-distributive.

Hence, we have the following:

Open problem 5.8 *Prove that any m-BCK algebra verifying (Pom) (i.e. taOM algebra) is a lattice w.r. to* $\leq_m$ *- or, equivalently, prove that any m-BCK algebra which is not a lattice does not verify (Pom) - or find an example of taOM algebra that is not a lattice. We have tried for several days to prove by Prover9 program [29] that any m-BCK algebra which is not a lattice, i.e. which has the situation from the Figure 7, does not verify (Pom).*

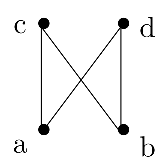


Figure 7: The Hasse diagram of the non-lattice situation of the order relation $\leq_m$

By using the following Mace4 input file, we have tried in vain, by Mace4 program [29], to find an example of finite taOM algebra that is not a lattice: practically, there is no such a finite example.

```
1 * x = x                                   #label ("PU").
x * y = y * x                               #label ("Pcomm").
x * (y * z) = (x * y) * z                   #label ("Pass").
x * -x = 0                                  #label ("m-Re").
x * 0 = 0                                   #label ("m-L").
- -x = x                                    #label ("DN").
-0 = 1                                      #label ("Neg0-1").
-1 = 0                                      #label ("Neg1-0").

((x * -y = 0) & (y * -x=0)) -> (x=y)        #label ("m-An").
(-(z * x) * (y * x)) * -(y * -z) = 0        #label ("m-BB").

-(-x * -y) = x + y                          #label ("sum").
-(-x + -y) = x * y                          #label ("product").
(x * y) + (-(x * y) * x) = x                #label("Pom").

a != 0.
a != 1.
b != 0.
b != 1.
c != 0.
c != 1.
d != 0.
d !=1.

c != d.
c != a.
c != b.
a != d.
a != b.
b != d.
(c * d) * -a = 0.
(c * d) * -b = 0.

a * -b != 0.
b * -a != 0.
c * -d !=0.
d * -c !=0.
```

a * -c = 0.
a * -d = 0.
b * -c = 0.
b * -d=0.
a * -x = 0 & x * -c = 0 − > (x=a | x=c).
a * -x = 0 & x * -d = 0 − > (x=a | x=d).
b * -x = 0 & x * -c = 0 − > (x=b | x=c).
b * -x = 0 & x * -d = 0 − > (x=b | x=d).
where != means "≠" and | means "or".

Open problem 5.9 *Prove that there is a connection between the open problem ([27], 4.14) connected to the aOM algebras and the previous open problem; most probably, there is an example of aOM algebra if and only if there is an example of taOM algebra that is not a lattice.*

We believe that there are no such examples, we believe that any taOM algebra is a lattice w.r.t. to $\leq_m$ and that we have: **MV** $\subset$ **taOM** $\subset$ **BCK-L**.

Remark 5.10 In a left-taOM algebra $\mathcal{A}^L = (A^L, \odot, ^-, 1)$:

- the initial binary relation, $\leq_m$ ($x \leq_m y \Longleftrightarrow x \odot y^- = 0$), is an order, since (m-Re), (m-An), (m-Tr) hold, namely a (distributive or not distributive) lattice order in all the finite examples found;

- the binary relation $\leq_m^M$ ($x \leq_m^M y \Longleftrightarrow x \wedge_m^M y = x$) is an order, by Corollary 3.7, but not a lattice order, in general, with respect to $\wedge_m^M$, $\vee_m^M$, since $x \wedge_m^M y \neq y \wedge_m^M x$; it is a distributive lattice order in MV algebras, where $x \wedge_m^M y = y \wedge_m^M x$ and $\leq_m^M \Longleftrightarrow \leq_m$;

- the binary relation $\leq_m^P$ ($x \leq_m^P y \Longleftrightarrow x \odot y = x$) is only antisymmetric and transitive, by ([25], Proposition 3.11); in Boolean algebras, it is a distributive lattice order and $\leq_m^P \Longleftrightarrow \leq_m \Longleftrightarrow \leq_m^M$.

Resuming, the examples we found helped us to conclude that we have the connections from the Figure 8.

This research is continued in [23, 24].

6 Examples

We introduce the following definition: an X algebra is said to be *proper*, if it verifies the properties from its definition and does not verify the other

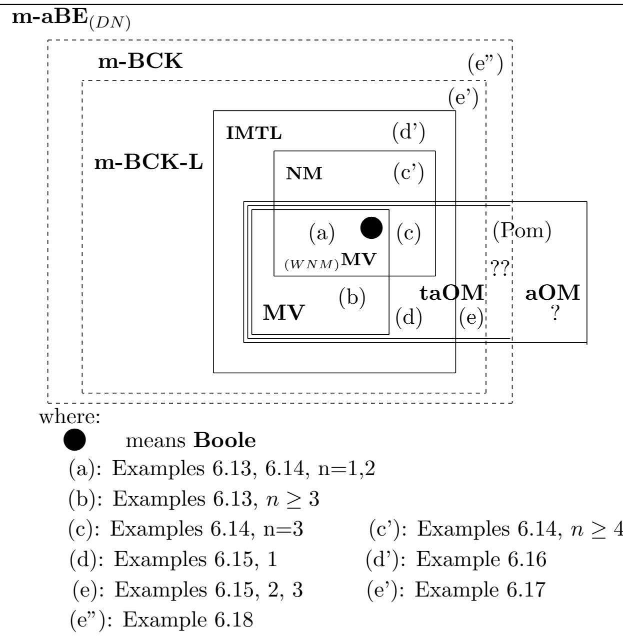


Figure 8: Resuming connections between **MV**, **NM**, **IMTL**, **taOM** and **aOM**, where ? means that there is an open problem concerning **aOM** and ?? means that there is an open problem concerning taOM non-lattices

properties (from this paper and from the previous papers, recalled below), except (prel$_m$) from [27] and (WNM$_m$), (aWNM$_m$) from [26].

(prel$_m$)	$(x \to y) \vee_m^B (y \to x) = 1$, where $x \to y \stackrel{def.}{=} (x \odot y^-)^-$,
(WNM$_m$)	$(x \odot y)^- \vee [(x \wedge y) \to (x \odot y)] = 1$,
	where $x \wedge y \stackrel{def.}{=} y \wedge_m^M x$, $x \vee y \stackrel{def.}{=} y \vee_m^M x$,
(aWNM$_m$)	$(x \odot y) \odot [x \odot (x \odot y^-)^- \odot (x \odot y)^-]^- = x \odot y$,
(m-Pimpl)	$[(x \odot y^-)^- \odot x^-]^- = x$ [22],
(G)	$x \odot x = x$ [22],
(m-Pabs-i)	$x \odot (x \oplus x \oplus y) = x$ [26],
(m-Pdis)	$z \odot (x \oplus y) = (z \odot x) \oplus (z \odot y)$ [25].

6.1 At Involutive m-BE Algebras Without (m-An) Level

Example 6.1 Proper involutive m-BE algebra (not verifying (trans)): m-BE$_{(DN)}$

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra not verifying (m-B) for (b,a,e), (m-BB) for (b,e,b), (m-*) for (a,e,b), (m-**) for (b,a,e), (m-Tr) for (b,a,e), (m-An) for (a,b), (m-Pimpl) for $(a,0)$, (m-Pabs-i) for $(d,0)$, (G) for a, (Pqmv) for $(b,0,c)$, (Pom) for (b,c), (Pmv) for (d,a), (Δ_m) for (a,d), (prel_m) for (e,f), (WNM_m) and (aWNM_m) for (e,b), (m-Pdis) for (a,a,b).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	a	0	0	0	a
b	0	0	0	f	0	a	0	b
c	0	a	f	c	0	e	f	c
d	0	0	0	0	0	0	0	d
e	0	0	a	e	0	0	a	e
f	0	0	0	f	0	a	0	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	e
f	f
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	1	1	1	f	b	1	1
b	b	1	1	1	b	1	1	1
c	c	1	1	1	1	1	1	1
d	d	f	b	1	d	e	f	1
e	e	b	1	1	e	1	b	1
f	f	1	1	1	f	b	1	1
1	1	1	1	1	1	1	1	1

.

The table of $\wedge_m^M$ is:

$\wedge_m^M$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	b	f	d	a	f	a
b	0	a	b	f	d	e	f	b
c	0	a	b	c	d	e	f	c
d	0	0	0	0	d	0	0	d
e	0	a	0	e	d	e	0	e
f	0	a	b	f	d	a	f	f
1	0	a	b	c	d	e	f	1

.

The binary relation $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a,e,c): $a \leq_m^M e$, $e \leq_m^M c$, but $a \not\leq_m^M c$, since $a \wedge_m^M c = f \neq a$.

Example 6.2 Involutive m-BE algebra verifying (trans): TRANS

By a PASCAL program, we found that the algebra $\mathcal{A}^L = (A_6 = \{0,a,b,c,d,1\}, \odot, ^-, 1)$, with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an *involutive left-m-BE algebra*, since (PU), (Pcomm), (Pass), (m-L), (m-Re), (DN) hold, while (m-B) does not hold for (b,a,b), (m-BB) for (b,b,a), (m-*) for (a,c,b), (m-**) for (b,a,b), (m-Tr) for (b,a,c), (m-An) for (a,b), (m-Pimpl) for $(a,0)$, (Pqmv) for $(b,0,b)$, (Pom) for (b,b), (Pmv) for (b,b), (Δ_m) for (b,c), (m-Pabs-i) for $(a,0)$, (G) for a, (prel_m) for

(b,c), (WNM$_m$) and (aWNM$_m$) for (c,c), (m-Pdis) for (a,a,c).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	0	0	b
c	0	0	0	a	a	c
d	0	0	0	a	a	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	d	1	1	1
b	b	d	d	1	1	1
c	c	1	1	d	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

.

Then, the tables of $\wedge_m^M$ and its dual, $\vee_m^M$, are the following:

$\wedge_m^M$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	a	a	a
b	0	a	b	a	a	b
c	0	a	0	c	d	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

and

$\vee_m^M$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	b	c	d	1
b	b	a	b	1	d	1
c	c	d	d	c	d	1
d	d	d	d	c	d	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive (hence (trans) is verified), hence $\leq_m^M$ is an order relation, by Corollary 2.10, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.3 Proper PreMV algebra (not verifying (trans) and (m-Tr)): PreMV

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_9 = \{0, a, b, c, d, e, f, g, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pmv) (hence (Δ_m)) and also (prel$_m$), and not verifying (m-B) for (b,g,b), (m-BB) for (b,b,g), (m-*) for (a,e,g), (m-**) for (b,g,b), (m-Tr) for (b,g,a), (m-An) for (a,e), (m-Pimpl) for $(a,0)$, (m-Pabs-i) for $(a,0)$, (G) for a, (Pqmv) for $(b,0,c)$, (Pom) for (b,c), (WNM$_m$) and (aWNM$_m$) for (b,b), (m-Pdis) for (a,a,b).

$\odot$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	a
b	0	0	c	a	e	0	c	0	b
c	0	0	a	0	0	0	a	0	c
d	0	0	e	0	0	0	a	0	d
e	0	0	0	0	0	0	0	0	e
f	0	0	c	a	a	0	c	e	f
g	0	0	0	0	0	0	e	0	g
1	0	a	b	c	d	e	f	g	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
g	g
1	0

, with

$\oplus$	0	a	b	c	d	e	f	g	1
0	0	a	b	c	d	e	f	g	1
a	a	d	1	f	b	d	1	1	1
b	b	1	1	1	1	1	1	1	1
c	c	f	1	1	1	b	1	1	1
d	d	b	1	1	1	b	1	1	1
e	e	d	1	b	b	d	1	f	1
f	f	1	1	1	1	1	1	1	1
g	g	1	1	1	1	f	1	1	1
1	1	1	1	1	1	1	1	1	1

.

The table of $\wedge_m^M$ is:

$\wedge_m^M$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	e	a	a	e	a	g	a
b	0	a	b	c	d	e	f	g	b
c	0	a	c	c	d	e	c	g	c
d	0	a	c	c	d	e	c	g	d
e	0	a	e	a	e	e	a	e	e
f	0	a	b	c	d	e	f	g	f
g	0	a	b	c	d	e	c	g	g
1	0	a	b	c	d	e	f	g	1

.

The binary relation $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a, c, b): $a \leq_m^M c$, $c \leq_m^M b$, but $a \not\leq_m^M b$, since $a \wedge_m^M b = e \neq a$.

Example 6.4 Proper MMV algebra (not verifying (Pmv), (m-Tr) and (trans)): MMV

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_{10} = \{0, a, b, c, d, e, f, g, h, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Δ_m) and not verifying (m-B) for (b, g, b), (m-BB) for (b, b, g), (m-*) for (a, g, h), (m-**) for (b, g, b), (m-Tr) for (b, g, a), (m-An) for (a, e), (m-Pimpl) for $(a, 0)$, (m-Pabs-i) for $(a, 0)$, (G) for a, (Pqmv) for $(b, 0, c)$, (Pom) for (b, c), (Pmv) for (g, h), (prel$_m$) for (g, h), (WNM$_m$) and (aWNM$_m$) for (b, b), (m-Pdis) for (a, a, b).

$\odot$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	0	0	a
b	0	0	c	a	e	0	c	0	0	b
c	0	0	a	0	0	0	a	0	0	c
d	0	0	e	0	0	0	a	0	0	d
e	0	0	0	0	0	0	0	0	0	e
f	0	0	c	a	a	0	c	0	0	f
g	0	0	0	0	0	0	0	0	a	g
h	0	0	0	0	0	0	0	a	0	h
1	0	a	b	c	d	e	f	g	h	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
g	g
h	h
1	0

, with

$\oplus$	0	a	b	c	d	e	f	g	h	1
0	0	a	b	c	d	e	f	g	h	1
a	a	d	1	f	b	d	1	1	1	1
b	b	1	1	1	1	1	1	1	1	1
c	c	f	1	1	1	b	1	1	1	1
d	d	b	1	1	1	b	1	1	1	1
e	e	d	1	b	b	d	1	1	1	1
f	f	1	1	1	1	1	1	1	1	1
g	g	1	1	1	1	1	1	1	b	1
h	h	1	1	1	1	1	1	b	1	1
1	1	1	1	1	1	1	1	1	1	1

.

The table of $\wedge_m^M$ is:

$\wedge_m^M$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	a	e	a	a	e	a	g	h	a
b	0	a	b	c	d	e	f	g	h	b
c	0	a	c	c	d	e	c	g	h	c
d	0	a	c	c	d	e	c	g	h	d
e	0	a	e	a	e	e	a	g	h	e
f	0	a	b	c	d	e	f	g	h	f
g	0	a	b	c	d	e	f	g	0	g
h	0	a	b	c	d	e	f	0	h	h
1	0	a	b	c	d	e	f	g	h	1

.

The binary relation $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a, c, b): $a \leq_m^M c$, $c \leq_m^M b$, but $a \not\leq_m^M b$, since $a \wedge_m^M b = e \neq a$.

Example 6.5 Proper involutive m-pre-BCK algebra (not verifying (trans) and (Δ_m)): m-pre-BCK$_{(DN)}$ (= m-tBE$_{(DN)}$)

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (m-BB) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-Tr)), and (prel$_m$), and not verifying (m-An) for (a, f), (m-Pimpl) for $(a, 0)$, (m-Pabs-i) for $(a, 0)$, (G) for a, (Pqmv) for $(b, 0, d)$, (Pom) for (b, d), (Pmv) for (c, a), (Δ_m) for (a, c), (WNM$_m$) and (aWNM$_m$) for (b, b), (m-Pdis) for (a, a, b).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	a	0	0	a
b	0	0	a	0	e	a	0	b
c	0	0	0	0	0	0	0	c
d	0	a	e	0	d	e	f	d
e	0	0	a	0	e	a	0	e
f	0	0	0	0	f	0	0	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	b	1	f	1	1	b	1
b	b	1	1	b	1	1	1	1
c	c	f	b	c	1	e	f	1
d	d	1	1	1	1	1	1	1
e	e	1	1	e	1	1	1	1
f	f	b	1	f	1	1	b	1
1	1	1	1	1	1	1	1	1

.

The table of $\wedge_m^M$ is:

$\wedge_m^M$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	c	f	a	f	a
b	0	a	b	c	e	e	f	b
c	0	0	0	c	0	0	0	c
d	0	a	b	c	d	e	f	d
e	0	a	b	c	e	e	f	e
f	0	a	a	c	f	a	f	f
1	0	a	b	c	d	e	f	1

.

The binary relation $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a, e, d): $a \leq_m^M e$, $e \leq_m^M d$, but $a \not\leq_m^M d$, since $a \wedge_m^M d = f \neq a$.

Example 6.6 Involutive m-pre-BCK algebra verifying (trans): tTRANS

By a PASCAL program, we found that the algebra

$$\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), and also (prel$_m$), (WNM$_m$), (aWNM$_m$), and not verifying (m-An) for (a, d), (m-Pimpl) for $(a, 0)$, (Pqmv) for $(b, a, 0)$, (Pom) for (c, a), (Pmv) for (b, a), (Δ_m) for (a, b), (m-Pabs-i) for $(b, 0)$, (G) for a, (m-Pdis) for (a, a, a).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	a	0	a
b	0	0	0	0	0	b
c	0	a	0	c	a	c
d	0	0	0	a	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	d	1	1	1
b	b	d	b	1	d	1
c	c	1	1	1	1	1
d	d	1	d	1	1	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive (hence (trans) is verified), hence $\leq_m^M$ is an order relation, by Corollary 2.10, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Example 6.7 Proper transitive PreMV algebra (not verifying (trans)): tPreMV

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pmv) (hence (Δ_m)) and (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), and also (prel$_m$), and not verifying (m-An) for (a, e), (m-Pimpl) for $(a, 0)$, (m-Pabs-i) for $(a, 0)$, (G) for a, (Pqmv) for $(b, 0, c)$, (Pom) for (b, c), (WNM$_m$) and (aWNM$_m$) for (b, b), (m-Pdis) for (a, a, b).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	c	a	e	0	c	b
c	0	0	a	0	0	0	a	c
d	0	0	e	0	0	0	a	d
e	0	0	0	0	0	0	0	e
f	0	0	c	a	a	0	c	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	d	1	f	b	d	1	1
b	b	1	1	1	1	1	1	1
c	c	f	1	1	1	b	1	1
d	d	b	1	1	1	b	1	1
e	e	d	1	b	b	d	1	1
f	f	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

.

Then, the table of $\wedge_m^M$ is the following:

$\wedge_m^M$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	e	a	a	e	a	a
b	0	a	b	c	d	e	f	b
c	0	a	c	c	d	e	c	c
d	0	a	c	c	d	e	c	d
e	0	a	e	a	e	e	a	e
f	0	a	b	c	d	e	f	f
1	0	a	b	c	d	e	f	1

.

Note that $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a, c, b).

Example 6.8 Proper tMMV algebra (not verifying (Pmv) and (trans)): tMMV

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_{12} = \{0, a, b, c, d, e, f, g, h, i, j, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Δ_m) and (m-BB) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-Tr)), and (prel$_m$), and not verifying (m-An) for (a, g), (m-Pimpl) for $(b, 0)$, (m-Pabs-i) for $(a, 0)$, (G) for a, (Pqmv) for $(a, 0, a)$, (Pom) for (a, a), (Pmv) for (a, a), (WNM$_m$) and (aWNM$_m$) for (c, c), (m-Pdis) for (a, a, a).

$\odot$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	g	0	g	0	g	0	g	0	0	g	a
b	0	0	b	i	d	d	i	0	b	i	d	b
c	0	g	i	e	0	a	d	g	f	d	g	c
d	0	0	d	0	0	0	0	0	d	0	0	d
e	0	g	d	a	0	g	0	g	d	0	g	e
f	0	0	i	d	0	0	d	0	i	d	0	f
g	0	g	0	g	0	g	0	g	0	0	g	g
h	0	0	b	f	d	d	i	0	b	i	d	h
i	0	0	i	d	0	0	d	0	i	d	0	i
j	0	g	d	g	0	g	0	g	d	0	a	j
1	0	a	b	c	d	e	f	g	h	i	j	1

,

$\oplus$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	a	b	c	d	e	f	g	h	i	j	1
a	a	a	1	c	j	j	c	a	1	c	j	1
b	b	1	h	1	h	1	h	1	h	h	1	1
c	c	c	1	1	1	1	1	c	1	1	1	1
d	d	j	h	1	f	c	b	e	h	h	c	1
e	e	j	1	1	c	c	1	j	1	1	c	1
f	f	c	h	1	b	1	h	c	h	h	1	1
g	g	a	1	c	e	j	c	a	1	c	j	1
h	h	1	h	1	h	1	h	1	h	h	1	1
i	i	c	h	1	h	1	h	c	h	b	1	1
j	j	j	1	1	c	c	1	j	1	1	c	1
1	1	1	1	1	1	1	1	1	1	1	1	1

and $(0, a, b, c, d, e, f, g, h, i, j, 1)^- = (1, b, a, d, c, f, e, h, g, j, i, 0)$.

The table of $\wedge_m^M$ is:

$\wedge_m^M$	0	a	b	c	d	e	f	g	h	i	j	1
0	0	0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	g	0	a	0	g	0	0	g	a
b	0	0	b	f	d	d	f	0	h	i	d	b
c	0	a	i	c	d	e	f	g	f	i	j	c
d	0	0	d	d	d	d	d	0	d	d	d	d
e	0	a	d	e	d	e	d	g	d	d	j	e
f	0	0	i	i	d	d	f	0	f	i	d	f
g	0	a	0	a	0	a	0	g	0	0	g	g
h	0	0	b	f	d	d	f	0	h	i	d	h
i	0	0	i	f	d	d	f	0	f	i	d	i
j	0	a	d	e	d	e	d	g	d	d	j	j
1	0	a	b	c	d	e	f	g	h	i	j	1

.

The binary relation $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a, e, c): $a \leq_m^M e$, $e \leq_m^M c$, but $a \not\leq_m^M c$, since $a \wedge_m^M c = g \neq a$.

Example 6.9 Proper orthomodular algebra: OM

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_5 = \{0, a, b, c, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pom), and also (prel$_m$), (WNM$_m$), (aWNM$_m$), and not verifying (m-B) for (a, c, a), (m-BB) for (a, a, c), (m-*) for (c, b, a), (m-**) for (a, c, a), (m-Tr) for (a, c, b), (m-An) for (a, c), (m-Pimpl) for $(a, 0)$, (m-Pabs-i) for $(b, 0)$, (G) for b, (Pqmv) for $(b, b, 0)$, (Pmv) for (b, b), (Δ_m) for (a, b), (m-Pdis) for (a, a, a).

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

and

x	x^-
0	1
a	b
b	a
c	c
1	0

, with

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	1	1	1	1
b	b	1	b	1	1
c	c	1	1	1	1
1	1	1	1	1	1

.

Example 6.10 Proper transitive OM algebra : tOM

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (Pom) and (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), and also (prel$_m$), and not verifying (m-An) for (c, e), (m-Pimpl) for $(b, 0)$, (m-Pabs-i) for $(a, 0)$, (G) for a, (Pqmv) for $(a, a, 0)$, (Pmv) for (a, a), (Δ_m) for (b, a), (WNM$_m$) and (aWNM$_m$) for (d, d), (m-Pdis) for (a, b, a).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	0	0	a
b	0	0	b	c	c	e	e	b
c	0	0	c	0	0	0	0	c
d	0	0	c	0	a	0	a	d
e	0	0	e	0	0	0	0	e
f	0	0	e	0	a	0	a	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	a	1	d	d	f	f	1
b	b	1	1	1	1	1	1	1
c	c	d	1	b	1	b	1	1
d	d	d	1	1	1	1	1	1
e	e	f	1	b	1	b	1	1
f	f	f	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

.

Then, the tables of $\wedge_m^M$ and its transposed, $\wedge_m^B$, are the following:

$\wedge_m^M$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	0	a	a
b	0	a	b	c	d	e	f	b
c	0	a	c	c	c	e	e	c
d	0	a	c	c	d	e	f	d
e	0	a	e	c	c	e	e	e
f	0	a	e	c	d	e	f	f
1	0	a	b	c	d	e	f	1

and

$\wedge_m^B$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a	a
b	0	0	b	c	c	e	e	b
c	0	0	c	c	c	c	c	c
d	0	a	d	c	d	c	d	d
e	0	0	e	e	e	e	e	e
f	0	a	f	e	f	e	f	f
1	0	a	b	c	d	e	f	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.7, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative. From the table of $\wedge_m^M$, we see that $a \leq_m^M d, f, 1$; $b \leq_m^M 1$; $c \leq_m^M b, d, 1$; $d \leq_m^M 1$; $e \leq_m^M b, f, 1$; $f \leq_m^M 1$; then, the bounded po-set $(A_8, \leq_m^M, 0, 1)$ is reprezented by the Hasse diagram from the Figure 9:

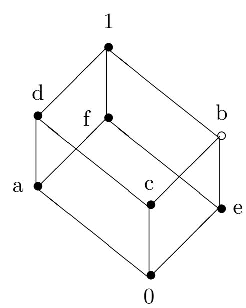


Figure 9: The Hasse diagram of the bounded po-set $(A_8, \leq_m^M, 0, 1)$

The binary relation $\leq_m$ ($\Longleftrightarrow \leq_m^B$) is a pre-order relation, since (m-Re) and (m-Tr) hold; hence, $\leq_m^B$ is a pre-order relation too. From the table of $\wedge_m^B$, we see that $a \leq_m b, c, d, e, f, 1$; $b \leq_m 1$; $c \leq_m b, d, e, f, 1$; $d \leq_m b, f, 1$; $e \leq_m b, c, d, f, 1$; $f \leq_m b, d, 1$; hence, the Hasse type diagram ([20], Remark 2.1.21) of the bounded, pre-ordered set $(A_8, \leq_m, 0, 1)$ is presented in the Figure 10.

6.2 At involutive m-aBE algebras level

Example 6.11 Proper involutive m-aBE algebra (not verifying (trans)): m-aBE$_{(DN)}$

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_{10} = \{0, a, b, c, d, e, f, g, h, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is an involutive left-m-BE algebra verifying (m-An) and not verifying (m-B) for (a, g, a), (m-BB) for (a, a, g), (m-*) for (a, g, c), (m-**) for (a, g, a), (m-Tr)

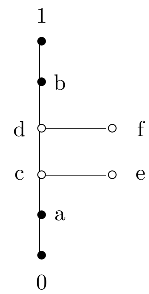


Figure 10: The Hasse type diagram of the bounded pre-ordered set $(A_8, \leq_m, 0, 1)$

for (a, g, b), (m-Pimpl) for $(a, 0)$, (m-Pabs-i) for $(a, 0)$, (G) for b, (Pqmv) for $(a, d, 0)$, (Pom) for (b, b), (Pmv) for (a, d), (Δ_m) for (a, b), (prel_m) for (a, b), (WNM_m) and (aWNM_m) for (b, b), (m-Pdis) for (a, a, a).

$\odot$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	a	0	a	0	a	0	0	0	a
b	0	0	g	g	0	d	0	d	d	b
c	0	a	g	e	0	a	d	d	f	c
d	0	0	0	0	0	0	0	0	0	d
e	0	a	d	a	0	a	0	0	d	e
f	0	0	0	d	0	0	0	0	d	f
g	0	0	d	d	0	0	0	0	0	g
h	0	0	d	f	0	d	d	0	f	h
1	0	a	b	c	d	e	f	g	h	1

,

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
g	h
h	g
1	0

,

$\oplus$	0	a	b	c	d	e	f	g	h	1
0	0	a	b	c	d	e	f	g	h	1
a	a	h	1	1	h	1	c	c	c	1
b	b	1	b	1	b	1	b	1	1	1
c	c	1	1	1	1	1	1	1	1	1
d	d	h	b	1	f	c	b	e	c	1
e	e	1	1	1	c	1	1	c	1	1
f	f	c	b	1	b	1	b	c	1	1
g	g	c	1	1	e	c	c	e	1	1
h	h	c	1	1	c	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1

.

The table of $\wedge_m^M$ is:

$\wedge_m^M$	0	a	b	c	d	e	f	g	h	1
0	0	0	0	0	0	0	0	0	0	0
a	0	a	d	f	d	a	f	d	f	a
b	0	0	b	g	d	d	f	g	h	b
c	0	a	b	c	d	e	f	g	h	c
d	0	0	d	d	d	d	d	d	d	d
e	0	a	b	e	d	e	f	g	f	e
f	0	0	g	g	d	d	f	g	f	f
g	0	a	g	a	d	a	d	g	d	g
h	0	a	g	e	d	e	f	g	h	h
1	0	a	b	c	d	e	f	g	h	1

.

The binary relation $\leq_m^M$ is not transitive (hence (trans) is not verified) for (a, e, c): $a \leq_m^M e$, $e \leq_m^M c$, but $a \not\leq_m^M c$, since $a \wedge_m^M c = f \neq a$.

Example 6.12 Proper involutive m-aBE algebra verifying (trans) and not verifying (m-Tr): aTRANS

By MACE4 program, we found that the algebra

$$\mathcal{A}^L = (A_7 = \{0, a, b, c, d, e, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper involutive left-m-aBE algebra, since it does not verify (m-B) for (b, c, b), (m-BB) for (b, b, c), (m-*) for (c, a, b), (m-**) for (b, c, b), (m-Tr) for (b, c, a), ($\wedge_m$-comm) for (a, e), (m-Pabs-i) for $(a, 0)$, (G) for a, (m-Pimpl) for $(a, 0)$, (Pqmv) for $(a, d, 0)$, (Pom) for (a, a), (Pmv) for (a, d), (Δ_m) for (c, b) and also (prel$_m$) for (a, b), (WNM$_m$) and (aWNM$_m$) for (a, a), (m-Pdis) for (a, a, a).

$\odot$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	c	0	d	0	c	a
b	0	0	d	0	0	d	b
c	0	d	0	0	0	d	c
d	0	0	0	0	0	0	d
e	0	c	d	d	0	c	e
1	0	a	b	c	d	e	1

and

x	x^-
0	1
a	b
b	a
c	c
d	e
e	d
1	0

, with

$\oplus$	0	a	b	c	d	e	1
0	0	a	b	c	d	e	1
a	a	e	1	1	e	1	1
b	b	1	c	e	c	1	1
c	c	1	e	1	e	1	1
d	d	e	c	e	c	1	1
e	e	1	1	1	1	1	1
1	1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive (hence (trans) is verified), hence $\leq_m^M$ is an order relation, by Corollary 2.10, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

Examples 6.13 Linearly ordered MV and $_{(WNM)}$MV algebras: MV and $_{(WNM)}$MV

Recall from ([17], 4.1.1) the following classes of examples of finite linearly ordered MV algebras and $_{(WNM)}$MV algebras.

Consider the linearly ordered (where $\leq \overset{notation}{=} \leq_m$) set (chain) $L_{n+1} = \{0, 1, 2, \ldots, n\}$, $(n \geq 1)$, organized as a lattice with $\wedge = \min$ and $\vee = \max$, and organized as left-MV algebra: $\mathcal{L}_{n+1} = (L_{n+1}, \odot, ^-, n)$, with:

$$x \odot y = \max\{0, x + y - n\},\ x^- = n - x = x \to 0, \quad 0 = n^-,$$

where: $x \to y = \max\{z \mid z \odot x \leq y\} = (x \odot y^-)^- = \min(n, y - x + n)$.

Hence, for $n = 1, 2, 3$, we have the linearly ordered left-MV algebras $\mathcal{L}_2$, $\mathcal{L}_3$, $\mathcal{L}_4$, whose tables are the following ($x^- = x \to 0$):

$$\mathcal{L}_2 \quad \begin{array}{c|cc} \odot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \text{ and } \begin{array}{c|c} x & x^- \\ \hline 0 & 1 \\ 1 & 0 \end{array}, \text{ with } \begin{array}{c|cc} \oplus & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}, \text{ and } \begin{array}{c|cc} \to & 0 & 1 \\ \hline 0 & 1 & 1 \\ 1 & 0 & 1 \end{array},$$

$$\mathcal{L}_3 \quad \begin{array}{c|ccc} \odot & 0 & 1 & 2 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 2 \end{array} \text{ and } \begin{array}{c|c} x & x^- \\ \hline 0 & 2 \\ 1 & 1 \\ 2 & 0 \end{array}, \text{ with } \begin{array}{c|ccc} \oplus & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{array}, \text{ and } \begin{array}{c|ccc} \to & 0 & 1 & 2 \\ \hline 0 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 2 \end{array},$$

$$\mathcal{L}_4 \quad \begin{array}{c|cccc} \odot & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 & 2 \\ 3 & 0 & 1 & 2 & 3 \end{array} \text{ and } \begin{array}{c|c} x & x^- \\ \hline 0 & 3 \\ 1 & 2 \\ 2 & 1 \\ 3 & 0 \end{array}, \text{ with } \begin{array}{c|cccc} \oplus & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 \end{array}, \text{ and } \begin{array}{c|cccc} \to & 0 & 1 & 2 & 3 \\ \hline 0 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 3 & 3 \\ 2 & 1 & 2 & 3 & 3 \\ 3 & 0 & 1 & 2 & 3 \end{array}.$$

Note that:

(1) For $n = 1, 2$, the MV algebras $\mathcal{L}_2$ and $\mathcal{L}_3$ verify condition (WNM) ($(x \odot y)^- \vee [(x \wedge y) \to (x \odot y)] = 1$), hence they are examples of ${}_{(WNM)}$MV algebras. Note that $\mathcal{L}_2$ is just the Boolean algebra with two elements.

(2) For $n = 3$, the MV algebra $\mathcal{L}_4$ does not verify condition (WNM) for 2. Hence, $\mathcal{L}_4$ is a proper MV algebra.

For $n \geq 4$, the MV algebra $\mathcal{L}_{n+1}$ does not verify condition (WNM) for $(n-2,\ n-1)$:

indeed, $[(n-2) \odot (n-1)]^- \vee [(n-2) \wedge (n-1) \to (n-2) \odot (n-1)]$ $= (n-3)^- \vee [(n-2) \to (n-3)] = 3 \vee (n-1) = n-1 \neq n$, because: $(n-2) \odot (n-1) = \max(0, (n-2)+(n-1)-n) = \max(0, n-3) = n-3$, since $n-3 \geq 4-3 = 1$, $(n-3)^- = n-(n-3) = 3$, $(n-2) \to (n-3) = \min(n, (n-3)-(n-2)+n) = \min(n, n-1) = n-1$ and $n-1 \geq 4-1 = 3$. Hence, $\mathcal{L}_{n+1}$ $(n \geq 3)$ is a proper MV algebra.

Examples 6.14 Linearly ordered NM algebras: NM

Recall from ([17], 5.1.1) the following classes of examples of linearly ordered (where $\leq \overset{notation}{=} \leq_m$) NM algebras and ${}_{(WNM)}$MV algebras.

For each $n \geq 1$, let us consider the chain $L_{n+1} = \{0, 1, 2, \ldots, n\}$, organized as a lattice w.r.t. to the lattice order $\leq = \leq_m$ and the lattice operations

$\wedge = \min$ and $\vee = \max$, and organized as an involutive residuated left-lattice $\mathcal{F}_{n+1} = (L_{n+1}, \wedge, \vee, \odot_F, \to_F, 0, n)$ in the following way: we take the strong negation $^-$, defined on L_{n+1} by $x^- = n - x$, and Fodor's implication $\to_F$ with the corresponding Fodor's t-norm $\odot_F$ [8], [6], defined by:

$$x \to_F y = \begin{cases} n, & \text{if } x \leq y \\ \max(n - x, y), & \text{if } x > y, \end{cases}$$

$$x \odot_F y = (x \to_F y^-)^- = \begin{cases} 0, & \text{if } x \leq n - y \\ \min(x, y), & \text{if } x > n - y. \end{cases}$$

Hence, for $n = 1, 2, 3, 4$, we have the involutive residuated left-lattices $\mathcal{F}_2$, $\mathcal{F}_3$, $\mathcal{F}_4$, $\mathcal{F}_5$, whose tables are the following (you have the values of $x^- = x \to_F 0$ in the table of $\to_F$, column of 0):

$\mathcal{F}_2$

$\to_F$	0	1
0	1	1
1	0	1

$\odot_F$	0	1
0	0	0
1	0	1

with

$\oplus_F$	0	1
0	0	1
1	1	1

,

$\mathcal{F}_3$

$\to_F$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

$\odot_F$	0	1	2
0	0	0	0
1	0	0	1
2	0	1	2

with

$\oplus_F$	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

,

$\mathcal{F}_4$

$\to_F$	0	1	2	3
0	3	3	3	3
1	2	3	3	3
2	1	**1**	3	3
3	0	1	2	3

$\odot_F$	0	1	2	3
0	0	0	0	0
1	0	0	0	1
2	0	0	**2**	2
3	0	1	2	3

with

$\oplus_F$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	3	3
3	3	3	3	3

,

$\mathcal{F}_5$

$\to_F$	0	1	2	3	4
0	4	4	4	4	4
1	3	4	4	4	4
2	2	**2**	4	4	4
3	1	**1**	**2**	4	4
4	0	1	2	3	4

$\odot_F$	0	1	2	3	4
0	0	0	0	0	0
1	0	0	0	0	1
2	0	0	0	**2**	2
3	0	0	**2**	**3**	3
4	0	1	2	3	4

with

$\oplus_F$	0	1	2	3	4
0	0	1	2	3	4
1	1	1	2	4	4
2	2	2	4	4	4
3	3	4	4	4	4
4	4	4	4	4	4

.

For each $n \geq 1$, $\mathcal{F}_{n+1} = (L_{n+1}, \wedge, \vee, \odot_F, \to_F, 0, 1)$ is an involutive residuated left-lattice that is linearly ordered, hence it satisfies condition

(prel) $((x \to_F y) \vee (y \to_F x) = 1)$;

they satisfy also condition

(WNM), thus, $\mathcal{F}_{n+1}$ is a NM algebra (Definition 1).

Note that:

(1) For $n = 1, 2$, $\mathcal{F}_2 = \mathcal{L}_2$ and $\mathcal{F}_3 = \mathcal{L}_3$, i.e. $\mathcal{F}_2$ and $\mathcal{F}_3$ are examples of linearly ordered $_{(WNM)}$MV algebras. Note that $\mathcal{F}_2 = \mathcal{L}_2$ is the Boolean algebra with two elements.

(2) For each $n \geq 3$, $\mathcal{F}_{n+1}$ is a linearly ordered proper NM algebra (i.e. not being MV algebra).

Consequently, for each $n \geq 1$, $\mathcal{F}^m_{n+1} = (L_{n+1}, \odot_F, ^-, n)$ is a NM algebra (Definition 2).

Note that:

(1') For $n = 1, 2$, since $\mathcal{F}^m_2 = \mathcal{L}_2$ and $\mathcal{F}^m_3 = \mathcal{L}_3$, it follows that $\mathcal{F}^m_2$ and $\mathcal{F}^m_3$ are NM algebras verifying (Pom) (since they are also MV algebras).

(2') For $n = 3$, $\mathcal{F}^m_4 = (L_4, \odot_F, ^-, 3)$ is a NM algebra verifying (Pom), hence **it is a proper taOM algebra** (i.e. not being MV algebra).

(3') For $n = 4$, $\mathcal{F}^m_5 = (L_5, \odot_F, ^-, 4)$ is a NM algebra which does not verify (Pom) for $(x, y) = (3, 2)$.

We shall prove that for any $n \geq 4$, there exists $(x, y) = (n-1, 2)$ such that the NM algebra $\mathcal{F}^m_{n+1} = (L_{n+1}, \odot_F, ^-, n)$ does not verify (Pom), where:

(Pom) $(x \odot_F y) \oplus_F ((x \odot_F y)^- \odot_F x) = x$ or, equivalently,

$$(x \odot_F y)^- \odot_F ((x \odot_F y)^- \odot_F x)^- = x^-.$$

Indeed, for any $n \geq 4$,

$$\begin{aligned}
&(x \odot_F y)^- \odot_F ((x \odot_F y)^- \odot_F x)^- \\
&= ((n-1) \odot_F 2)^- \odot_F (((n-1) \odot_F 2)^- \odot_F (n-1))^- \\
&= 2^- \odot_F (2^- \odot_F (n-1))^- \\
&= (n-2) \odot_F ((n-2) \odot_F (n-1))^- \\
&= (n-2) \odot_F (n-2)^- \\
&= (n-2) \odot_F 2 = 0,
\end{aligned}$$

while $x^- = (n-1)^- = 1$;

since $0 \neq 1$, it follows that $(x \odot_F y)^- \odot_F ((x \odot_F y)^- \odot_F x)^- \neq x^-$, i.e. (Pom) does not hold for $(x, y) = (n-1, 2)$.

Hence, for $n \geq 4$, $\mathcal{F}^m_{n+1} = (L_{n+1}, \odot_F, ^-, n)$ is a proper NM algebra (i.e. not being taOM algebra).

Examples 6.15 Proper transitive, antisymmetric OM algebras: taOM

• **Example 1: taOM is IMTL**

By a PASCAL program, we found that the algebra

$$\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper transitive, antisymmetric left-orthomodular algebra (taOM) (= m-BCK algebra verifying (Pom)), i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An), (DN), (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), (Pom), but also (prel$_m$), hold and it does not verify (m-Pabs-i) for $(d, 0)$, (G) for b, (m-Pimpl) for $(a, 0)$, (Pqmv) for $(c, d, 0)$, (Pmv) for (c, d), (Δ_m) for (a, c), (WNM$_m$) and (aWNM$_m$) for (c, c).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	b	0	a
b	0	b	0	0	0	b
c	0	b	0	d	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	a	1	c	1
c	c	1	1	1	c	1
d	d	1	c	c	d	1
1	1	1	1	1	1	1

.

Then, the tables of $\wedge_m^M$ and its transposed, $\wedge_m^B$, are the following:

$\wedge_m^M$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	c	d	a
b	0	b	b	b	d	b
c	0	b	b	c	d	c
d	0	0	0	d	d	d
1	0	a	b	c	d	1

and

$\wedge_m^B$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	b	0	a
b	0	b	b	b	0	b
c	0	c	b	c	d	c
d	0	d	d	d	d	d
1	0	a	b	c	d	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.7, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative. From the table of $\wedge_m^M$, we see that $a \leq_m^M 1$; $b \leq_m^M a, c, 1$; $c \leq_m^M 1$; $d \leq_m^M c, 1$; hence, the bounded po-set $(A_6, \leq_m^M, 0, 1)$ is reprezented by the Hasse diagram from the Figure 11.

The binary relation $\leq_m$ ($\Longleftrightarrow \leq_m^B$) is an order relation also, since (m-Re), (m-An) and (m-Tr) hold; hence, $\leq_m^B$ is an order relation too. From the table of $\wedge_m^B$, we see that $0 \leq_m d \leq_m b \leq_m c \leq_m a \leq_m 1$, hence the binary relations $\leq_m^B \Longleftrightarrow \leq_m$ ($x \leq_m^B y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^B y = x$) are linearly ordered. Note that the operation $\wedge_m^B$ is not commutative, therefore the order relation $\leq_m^B$ is not a lattice order w.r. to $\wedge_m^B$, $\vee_m^B$; but, the order relation $\leq_m$ is a lattice order

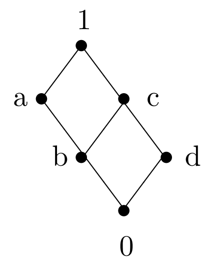


Figure 11: The Hasse diagram of the bounded po-set $(A_6, \leq_m^M, 0, 1)$

w.r. to $\wedge = \wedge_m = \inf_m$, $\vee = \vee_m = \sup_m$:

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	b	c	d	a
b	0	b	b	b	d	b
c	0	c	b	c	d	c
d	0	d	d	d	d	d
1	0	a	b	c	d	1

and

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	a	a	a	1
b	b	a	b	c	b	1
c	c	a	c	c	c	1
d	d	a	b	c	d	1
1	1	1	1	1	1	1

.

To see if the properties (prel) ($(x \to y) \vee (y \to x) = 1$) and (WNM) ($(x \odot y)^- \vee [(x \wedge y) \to (x \odot y)] = 1$) are verified, we need the table of $\to$ ($x \to y \overset{def.}{=} (x \odot y^-)^-$):

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	c	c	d	1
b	c	1	1	1	c	1
c	b	1	a	1	c	1
d	a	1	1	1	1	1
1	0	a	b	c	d	1

.

It is easy to see that (prel) is verified. We check by a PASCAL program that (WNM) is not verified for (a, b), (c, b), (c, d). Hence, this taOM algebra is an IMTL algebra.

Note that, by denoting $(0, 1, 2, 3, 4, 5) = (0, d, b, c, a, 1)$, we obtain that this IMTL algebra is one of the two linearly ordered IMTL algebras with 6 elements verifying (Pom) from ([17], 5.1.1): IMTL_6^4, where $0 \leq_m 1 \leq_m 2 \leq_m 3 \leq_m 4 \leq_m 5$; the other one is IMTL_6^3.

• **Example 2: taOM is a distributive lattice**

By Mace4 program, we found that the algebra

$$\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper transitive, antisymmetric left-orthomodular algebra (taOM) (= m-BCK algebra verifying (Pom)), i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An), (DN), (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), (Pom) hold and it does not verify (m-Pabs-i) for $(b, 0)$, (G) for b, (m-Pimpl) for $(a, 0)$, (Pqmv) for $(b, b, 0)$, (Pmv) for (b, b), (Δ_m) for (a, b), (prel$_m$) for (b, a), (WNM$_m$) and (aWNM$_m$) for (b, b).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
b	0	0	c	0	c	b
c	0	0	0	0	0	c
d	0	a	c	0	a	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	d	1	d	1	1
b	b	1	b	b	1	1
c	c	d	b	b	1	1
d	d	1	1	1	1	1
1	1	1	1	1	1	1

.

Then, the tables of $\wedge_m^M$ and its transposed, $\wedge_m^B$, are the following:

$\wedge_m^M$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	a	a
b	0	0	b	c	c	b
c	0	0	c	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

and

$\wedge_m^B$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	0	a	a
b	0	c	b	c	b	b
c	0	c	c	c	c	c
d	0	a	c	c	d	d
1	0	a	b	c	d	1

.

Note that $\leq_m^M$ is an order relation, by Corollary 3.7, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative. From the table of $\wedge_m^M$, we see that $a \leq_m^M d, 1$; $b \leq_m^M 1$; $c \leq_m^M b, d, 1$; $d \leq_m^M 1$; hence, the bounded po-set $(A_6, \leq_m^M, 0, 1)$ is reprezented by the Hasse diagram from the Figure 12:

The binary relation $\leq_m$ ($\Longleftrightarrow \leq_m^B$) is an order relation also, since (m-Re), (m-An) and (m-Tr) hold; hence, $\leq_m^B$ is an order relation too. From the table of $\wedge_m^B$, we see that $a \leq_m d, 1$; $b \leq_m d, 1$; $c \leq_m a, b, d, 1$; $d \leq_m 1$; hence, the binary relations $\leq_m^B \Longleftrightarrow \leq_m$ ($x \leq_m^B y \overset{def.}{\Longleftrightarrow} x \wedge_m^B y = x$) are represented by the Hasse diagram from the Figure 13.

Note that the operation $\wedge_m^B$ is not commutative, therefore the order relation $\leq_m^B$ is not a lattice order w.r. to $\wedge_m^B$, $\vee_m^B$; but, the order relation

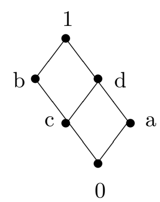


Figure 12: The Hasse diagram of the bounded po-set $(A_6, \leq_m^M, 0, 1)$

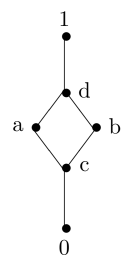


Figure 13: The Hasse diagram of the bounded po-set $(A_6, \leq_m, 0, 1)$

$\leq_m$ is a distributive lattice order w.r.t. to $\wedge = \wedge_m = \inf_m$, $\vee = \vee_m = \sup_m$:

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	a	a
b	0	c	b	c	b	b
c	0	c	c	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

and

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	d	a	d	1
b	b	d	b	b	d	1
c	c	a	b	c	d	1
d	d	d	d	d	d	1
1	1	1	1	1	1	1

.

To see if the property (prel) $((x \to y) \vee (y \to x) = 1)$ is verified, we need the table of $\to$ $(x \to y \overset{def.}{=} (x \odot y^-)^-)$:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	b	1	1
b	a	d	1	d	1	1
c	d	1	1	1	1	1
d	c	d	b	b	1	1
1	0	a	b	c	d	1

.

It is easy to see that (prel) is not verified for (a, b): $(a \to b) \vee (b \to a) =$

$b \vee d = d \neq 1$. Hence, this taOM algebra is not an IMTL algebra, it is only a distributive lattice.

• **Example 3: taOM is a non-distributive lattice (Michael Kinyon)**

By Mace4 program, Michael Kinyon found that the algebra

$$\mathcal{A}^L = (A_8 = \{0, a, b, c, d, e, f, 1\}, \odot, ^-, 1),,$$

with the following tables of $\odot$ and $^-$, is a proper transitive, antisymmetric left-orthomodular algebra (taOM) (= m-BCK algebra verifying (Pom)), verifying (prel$_m$), (WNM$_m$) and (aWNM$_m$), not verifying (m-Pabs-i) for $(b, 0)$, (G) for b, (m-Pimpl) for $(a, 0)$, (Pqmv) for $(b, b, 0)$, (Pmv) for (b, b), (Δ_m) for (a, b), (m-Pdis) for (a, a, a).

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	0	a	a
b	0	0	c	c	e	0	c	b
c	0	0	c	c	0	0	c	c
d	0	a	e	0	a	0	a	d
e	0	0	0	0	0	0	0	e
f	0	a	c	c	a	0	f	f
1	0	a	b	c	d	e	f	1

and

x	x^-
0	1
a	b
b	a
c	d
d	c
e	f
f	e
1	0

, with

$\oplus$	0	a	b	c	d	e	f	1
0	0	a	b	c	d	e	f	1
a	a	d	1	f	d	d	1	1
b	b	1	b	b	1	b	1	1
c	c	f	b	b	1	b	1	1
d	d	d	1	1	d	d	1	1
e	e	d	b	b	d	e	1	1
f	f	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

.

The tables of $\wedge_m^B$ and $\rightarrow$ are:

$\wedge_m^B$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	0	0	a	0	a	a
b	0	e	b	c	e	e	b	b
c	0	0	c	c	0	0	c	c
d	0	a	e	e	d	e	d	d
e	0	e	e	e	e	e	e	e
f	0	a	c	c	a	0	f	f
1	0	a	b	c	d	e	f	1

and

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	b	1	b	b	1	b	1	1
b	a	d	1	f	d	d	1	1
c	d	d	1	1	d	d	1	1
d	c	f	b	b	1	b	1	1
e	f	1	1	1	1	1	1	1
f	e	d	b	b	d	e	1	1
1	0	a	b	c	d	e	f	1

.

From the table of $\wedge_m^B$, we can see easily that: $a \leq_m d, f, 1$; $b \leq_m f, 1$; $c \leq_m b, f, 1$; $d \leq_m f, 1$; $e \leq_m a, b, c, d, f, 1$; $f \leq_m 1$; it follows that the Hasse diagram of the bounded po-set $(A_8, \leq_m, 0, 1)$ is that from the Figure 14.

The lattice is not distributive (it contains as sublattice the pentagon e, a, c, b, f) for (b, a, c): $b \wedge_m (a \vee_m c) = b \wedge_m f = b \neq (b \wedge_m a) \vee_m (b \wedge_m c) = e \vee_m c = c$.

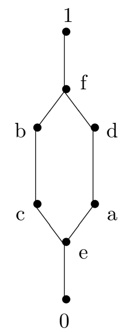


Figure 14: The Hasse diagram of the bounded po-set $(A_8, \leq_m, 0, 1)$, which is a non-distributive lattice

From the table of $\to$ we can see that (prel) is not satisfied for (b, d): $(b \to d) \vee (d \to b) = d \vee b = f \neq 1$. Hence, this taOM algebra is not an IMTL algebra, it is only a non-distributive lattice.

Example 6.16 Proper IMTL algebra (not verifying (Pom) and (WNM)): IMTL

Consider the following linearly ordered IMTL algebra (Definition 2)

$$\mathcal{A}^L = (A_5 = \{0, a, b, c, 1\}, \odot, ^-, 1),$$

with five elements, IMTL$_5$, from ([17], 5.1.1), where the tables of $\odot$ and $^-$, and of the additional operations $\oplus$ are the following.

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	a	b
c	0	0	a	a	c
1	0	a	b	c	1

and

x	x^-
0	1
a	c
b	b
c	a
1	0

, with

$\oplus$	0	a	b	c	1
0	0	a	b	c	1
a	a	c	c	1	1
b	b	c	1	1	1
c	c	1	1	1	1
1	1	1	1	1	1

.

Note that $\mathcal{A}^L$ is a proper left-m-BCK algebra, verifying (prel$_m$), not verifying (m-Pabs-i) for $(a, 0)$, (G) for a, (m-Pimpl) for $(a, 0)$, (Pqmv) for $(b, a, 0)$, (Pom) for (b, c), (Pmv) for (b, a), (Δ_m) for (c, b), (m-Pdis) for (a, a, b), (WNM$_m$) and (aWNM$_m$ for (b, c).

The tables of $\wedge^B$ and $\to$ are:

$\wedge_m^B$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	b	b
c	0	a	a	c	c
1	0	a	b	c	1

and

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	1	1
b	b	c	1	1	1
c	a	c	c	1	1
1	0	a	b	c	1

.

From the table of $\wedge_m^B$ ($x \leq_m y \Longleftrightarrow x \wedge_m^B y = x$), we see that $a \leq_m b, c, 1$; $b \leq_m c, 1$; $c \leq_m 1$; hence, we obtain the chain $0 \leq_m a \leq_m b \leq_m c \leq_m 1$. Thus, the bounded po-set $(A_5, \leq_m, 0, 1)$ is a distributive lattice, with $x \wedge_m y = \inf_m(x, y)$, $x \vee_m y = \sup_m(x, y)$. Hence, the left-m-BCK algebra is a lattice.

Note, from the table of $\rightarrow$, that (prel) is satisfied, but (WNM) is not satisfied for (b, c); $(b \odot c)^- \vee_m ((b \wedge_m c) \rightarrow (b \odot c)) = a^- \vee_m (b \rightarrow a) = c \vee_m c = c \neq 1$; hence, $\mathcal{A}^L$ is a left-IMTL algebra, not verifying (Pom) and (WNM), hence it is a proper IMTL algebra.

Example 6.17 Proper (involutive) m-BCK lattice: m-BCK-L

By a PASCAL program, we found that the algebra

$$\mathcal{A}^L = (A_6 = \{0, a, b, c, d, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$ and of the additional operation $\oplus$, is a proper left-m-BCK algebra, i.e. (PU), (Pcomm), (Pass), (m-L), (m-Re), (m-An), (DN) and (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)) hold and it does not verify (m-Pabs-i) for $(b, 0)$, (G) for a, (m-Pimpl) for $(a, 0)$, ($\wedge_m$-comm) for (a, b), (Pqmv) for $(b, d, 0)$, (Pom) for (b, a), (Pmv) for (b, d), (Δ_m) for (a, b) and also (prel$_m$) for (b, c), (WNM$_m$) and (aWNM$_m$) for (a, a), (m-Pdis) for (a, a, b).

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	d	d	0	a
b	0	d	d	0	0	b
c	0	d	0	d	0	c
d	0	0	0	0	0	d
1	0	a	b	c	d	1

and

x	x^-
0	1
a	d
b	c
c	b
d	a
1	0

, with

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	1	1	1	1	1
b	b	1	a	1	a	1
c	c	1	1	a	a	1
d	d	1	a	a	a	1
1	1	1	1	1	1	1

.

Note that $\leq_m^M$ is transitive, by Theorem 4.3, hence $\leq_m^M$ is an order relation, by Corollary 2.10, but not a lattice order w.r. to $\wedge_m^M$, $\vee_m^M$, since $\wedge_m^M$ is not commutative.

The binary relation $\leq_m$ ($\Longleftrightarrow \leq_m^B$) is an order relation also, since (m-Re), (m-An) and (m-Tr) hold; hence, $\leq_m^B$ is an order relation too. The tables of $\wedge_m^B$ and $\rightarrow$ are:

$\wedge_m^B$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	**d**	**d**	d	a
b	0	b	b	d	d	b
c	0	c	d	c	d	c
d	0	d	d	d	d	d
1	0	a	b	c	d	1

and

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	a	a	1
b	c	1	1	a	a	1
c	b	1	a	1	a	1
d	a	1	1	1	1	1
1	0	a	b	c	d	1

.

From the table of $\wedge_m^B$, we see that: $a \leq_m 1$; $b \leq_m a, 1$; $c \leq_m a, 1$; $d \leq_m a, b, c, 1$. It follows that the Hasse diagram of $\leq_m^B$ $(\Longleftrightarrow \leq_m)$ $(x \leq_m^B y \stackrel{def.}{\Longleftrightarrow} x \wedge_m^B y = x)$ is that from the Figure 15.

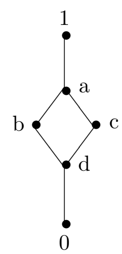


Figure 15: The Hasse diagram of the bounded po-set $(A_6, \leq_m, 0, 1)$, that is a distributive lattice

Note that the operation $\wedge_m^B$ is not commutative, therefore the order relation $\leq_m^B$ is not a lattice order w.r. to $\wedge_m^B$, $\vee_m^B$; but, the order relation $\leq_m$ is a distributive lattice order w.r. to $\wedge = \wedge_m = \inf_m$, $\vee = \vee_m = \sup_m$.

From the table of $\rightarrow$, note that $(b \rightarrow c) \vee (c \rightarrow b) = a \vee a = a \neq 1$, hence (prel) is not verified; hence, $\mathcal{A}^L$ is not an IMTL algebra. Hence, the algebra $\mathcal{A}^L$ is a proper (involutive) distributive left-m-BCK lattice.

Example 6.18 Proper m-BCK algebra (not lattice): m-BCK

Consider the algebra

$$\mathcal{A}^L = (A_8 = \{0, m, a, b, c, d, n, 1\}, \odot, ^-, 1),$$

with the following tables of $\odot$ and $^-$, from ([17], 7.2.1). It is an involutive left-m-BE algebra verifying (m-An) and (m-Tr) ($\Leftrightarrow \ldots \Leftrightarrow$ (m-BB)), and not verifying (m-Pimpl) for $(m, 0)$, (m-Pabs-i) for $(m, 0)$, (G) for m, (Pqmv) for $(a, m, 0)$, (Pom) for $(a, 0)$, (Pmv) for (a, m), (Δ_m) for (c, a), (prel_m) for (a, b), (WNM_m) and (aWNM_m) for (a, d). Hence, $\mathcal{A}^L$ is a left-m-BCK algebra.

$\odot$	0	m	a	b	c	d	n	1
0	0	0	0	0	0	0	0	0
m	0	0	0	0	0	0	0	m
a	0	0	0	0	0	m	m	a
b	0	0	0	0	m	0	m	b
c	0	0	0	m	m	m	m	c
d	0	0	m	0	m	m	m	d
n	0	0	m	m	m	m	m	n
1	0	m	a	b	c	d	n	1

,

x	x^-
0	1
m	n
a	c
b	d
c	a
d	b
n	m
1	0

,

$\oplus$	0	m	a	b	c	d	n	1
0	0	m	a	b	c	d	n	1
m	m	n	n	n	n	n	1	1
a	a	n	n	n	1	n	1	1
b	b	n	n	n	n	1	1	1
c	c	n	1	n	1	1	1	1
d	d	n	n	1	1	1	1	1
n	n	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1

.

To see if the m-BCK algebra is lattice or not w.r. to the order relation $\leq_m$ ($\Longleftrightarrow \leq_m^B$), we make the table of $\wedge_m^B$:

$\wedge_m^B$	0	m	a	b	c	d	n	1
0	0	0	0	0	0	0	0	0
m	0	m	m	m	m	m	m	m
a	0	m	a	m	a	a	a	a
b	0	m	m	b	b	b	b	b
c	0	m	m	m	c	m	c	c
d	0	m	m	m	m	d	d	d
n	0	m	m	m	m	m	n	n
1	0	m	a	b	c	d	n	1

.

From the table of $\wedge_m^B$, we see that $m \leq_m a, b, c, d, n, 1$; $a \leq_m c, d, n, 1$; $b \leq_m c, d, n, 1$; $c \leq_m n, 1$; $d \leq_m n, 1$; $n \leq_m 1$. Hence, the Hasse diagram of the bounded po-set $(A_8, \leq_m, 0, 1)$ is that from the Figure 16.

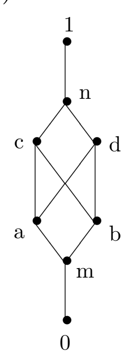


Figure 16: The Hasse diagram of the bounded po-set $(A_8, \leq_m, 0, 1)$, which is not a lattice

Note that the po-set $(A_8, \leq_m, 0, 1)$ is not a lattice. Hence, $\mathcal{A}^L$ is a proper m-BCK algebra, which is not a lattice.

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