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Topological Characterization of the Third Type of Triangular Hex-derived Networks

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Abstract

A topological index is a numerical quantity that defines a chemical descriptor to report several physical, biological and chemical properties of a chemical structure. In recent literature, various degree-based topological indices of a molecular structure are easily calculated by deriving a M-polynomial of that structure. In this paper, we first determine the expression of a M-polynomial of the triangular Hex-derived network of type three of dimension n and then obtain the corresponding degree-based topological indices from the closed form of M-polynomial. In addition, we use Maple software to represent the M-polynomial and the concerned degree-based topological indices pictorially for different dimensions.

Keywords: Third type of triangular Hex-derived network, Degree-based topological indices, M-polynomial, Graph polynomial.

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1 Introduction

Let us assume that $G = (V, E)$ be a simple, undirected and connected graph, where $V = V(G)$ indicates the vertex set and $E = E(G)$ indicates the edge set which consists of unordered pairs of vertices. We use the notation $d(u)$ to denote the *degree* of a vertex $u \in V(G)$ in G, which is the total number of vertices that are adjacent to the vertex u [1, 27, 38]. A network consists of nodes connected by links. While modeling a network mathematically, it is represented by a graph where the vertices of the graph are correlated to nodes and edges are correlated to the links between nodes of that network. In the scientific literature, both the terms[3] are used interchangeably.

A Short Note on Topological Indices and M-polynomials

In the area of Chemical Graph Theory (CGT) which is a branch of mathematical chemistry, the mathematical side of molecular compounds and their behaviours are being investigated. In CGT, a chemical compound is represented by a graph where the vertices of the graph correspond to atoms and edges correspond to the chemical bonds between the atoms of that chemical compound. For more detailed discussions about the chemical applications of graphs, please refer to [3, 10, 13, 17]. One can investigate several Physico-chemical characteristics of a chemical compound from its graph theoretical representation.

In the branch of CGT, a *topological index* (also known as a *connectivity index or graph-theoretic index*) stipulates several Physico-chemical and biological properties (such as strain energy, boiling point, fracture toughness, viscosity, heat of formation, etc.) of a chemical molecular structure. Essentially, a topological index is a numerical descriptor of a molecular structure, which is utilized in the study and research of Quantitative Structure Property Relationships (QSPRs) and Quantitative Structure Activity Relationships (QSARs)[4] [21, 35].

There are mainly four classes of topological indices, namely distance-based topological indices [4], degree-based topological indices [16], degree and distance-based topological indices [28] and counting-related topological indices [19]. In general, long computation is necessary to compute

[3]The term network is frequently used to refer to real-world systems, whereas the term graph is used when discussing the mathematical representation of the network.

[4]In the area of Cheminformatics, QSAR and QSPR are used to predict the chemical, physical and biological properties of a chemical compound.

the numeric values of the topological indices if we use their respective definitions. To simplify these computational procedures, the idea of special algebraic polynomials [15] is proposed in graph theory, from where we can derive several topological indices of a given structure by differentiating or integrating or both.

Among various such graph polynomials proposed in the literature, the M-polynomial is to used to simplify the calculation of nine degree-based topological indices of a chemical structure and it is proposed by Deutsch and Klavžar [9] in 2015. And hence the M-polynomials corresponding to various chemical networks have been studied extensively for various chemical networks, such as, polyhex nanotubes [24], V-phenylenic nanotubes and nanotori [20], Hex-derived network of type 1 and type 2 [18], convex polytopes [32], Hex-derived network [7], chain Hex-derived network [6, 29], etc. Please refer to their references also for more related works.

Definition 1.1 ([9]) *The M-polynomial of a simple connected graph G is given by the following expression:*

$$M(G;x,y) = \sum_{\delta \le i \le j \le \Delta} m_{i,j}(G)\ x^i y^j,$$

where $\delta = min\{d(u) | u \in V(G)\}$*,* $\Delta = max\{d(u) | u \in V(G)\}$*, and* $m_{i,j}(G)$ *is the number of edges* $uv \in E(G)$ *such that* $d(u) = i$ *and* $d(v) = j$*,* $(i, j \ge 1)$*.*

As cited in [8], in general, the degree-based topological index of a graph G is denoted as $I(G)$ and is given by

$$I(G) = \sum_{uv \in E(G)} f(d(u), d(v)) = \sum_{i \le j} m_{i,j}(G) f(i,j).$$

where the function $f(x, y)$ is differently chosen for different standard topological indices as given in Table 1.

Theorem 1.2 ([9], Theorems 2.1, 2.2) *Let us assume that G be a simple and connected graph.*

1. *If* $I(G) = \sum_{uv \in E(G)} f(d(u), d(v))$*, where* $f(x, y)$ *is a polynomial in* x *and* y*, then*

$$I(G) = f(D_x, D_y)(M(G;x,y))|_{x=y=1}.$$

2. *If* $I(G) = \sum_{uv \in E(G)} f(d(u), d(v))$*, where* $f(x, y) = \sum_{i,j \in \mathbb{Z}} \alpha_{i,j} x^i y^j$*, then* $I(G)$ *can be obtained from* $M(G; x, y)$ *using the operators* $D_x, D_y, S_x,$ *and* S_y*.*

3. *If* $I(G) = \sum_{uv \in E(G)} f(d(u), d(v))$*, where* $f(x, y) = \frac{x^r y^s}{(x+y+\alpha)^t}$*, where* $r, s \geq 0$*,* $t \geq 1$ *and* $\alpha \in \mathbb{Z}$*, then*

$$I(G) = S_x^t Q_\alpha J D_x^r D_y^s (M(G; x, y))|_{x=1}.$$

Standard Degree-based Topological Indices

Here, we brief some of the standard degree-based topological indices of our interest. Gutman and Trinajstić [14] introduced the *Zagreb indices* in 1972. They are useful in determining the total π-electron energy which is related to the thermodynamic stability of a given molecule. In the calculation of the Zagreb indices, higher priorities to the inner edges and vertices are given rather than the outer edges and vertices. With the opposite intuition and inspired by the idea of the Zagreb indices, the *modified Zagreb indices* are presented in [23]. Another very well-known degree-based topological index, with wide applications in the field of drug design, is the *Randić index* (*branching index* or *connectivity index*). It is proposed by Milan Randić [31] in 1975. Seeing the success of Randić index, after two decades, in 1998 the mathematicians Bollobás and Erdős [5] and Amić et al. [2] proposed the concept of a generalized version of the Randić index[5] (for an arbitrary real parameter α), which is known as *general Randić index*. In order to determine the total surface area of polychlorobiphenyls, the idea of the *symmetric division* (*deg*) *index* is presented around 2010 [36]. The *inverse sum* (*indeg*) *index* [33, 36] is for predicting the total surface area of octane isomers. To investigate the heat of formation of alkanes, the *augmented Zagreb index* is proposed by Furtula et al. in [12]. Table 1 lists all the formulas of different standard degree-based topological indices for a simple connected graph G.

[5] For $\alpha = -\frac{1}{2}$, $\alpha = 1$ and $\alpha = -1$, R_α becomes the Randić [31] (or connectivity) index, the second Zagreb index and the modified second Zagreb index, respectively.

The Triangular Hex-derived Networks of Type 3 of Dimension n

The n-dimensional Hexagonal network (denoted by $HX[n]$) and its properties are discussed in [26]. It has wide applications in image processing to model the wireless sensor networks, benzenoid hydrocarbons in the field of chemistry and computer graphics. Further in 2008, the constructions of Hex-derived network of type 1 ($HDN1$) and type 2 ($HDN2$) are presented in [22], having several applications in electronics, pharmaceutical sciences, and communication networks. Later in 2017, Raj and George [30] proposed new chemical networks which are derived from $HX[n]$, namely, Hex-derived networks of type 3 of dimension n ($HDN3[n]$), chain Hex-derived network of type 3 of dimension n ($CHDN3[n]$) and poly Hex-derived networks of type 3 of dimension n ($PHDN3[n]$) that consists of triangular Hex-derived network of type 3 of dimension n ($THDN3[n]$) and rectangular Hex-derived network of type 3 of dimension n ($RHDN3[n]$). A pictorial example of a third type triangular Hex-derived network of dimension 7 ($THDN3[7]$) is shown in Figure 1 (in Page 149).

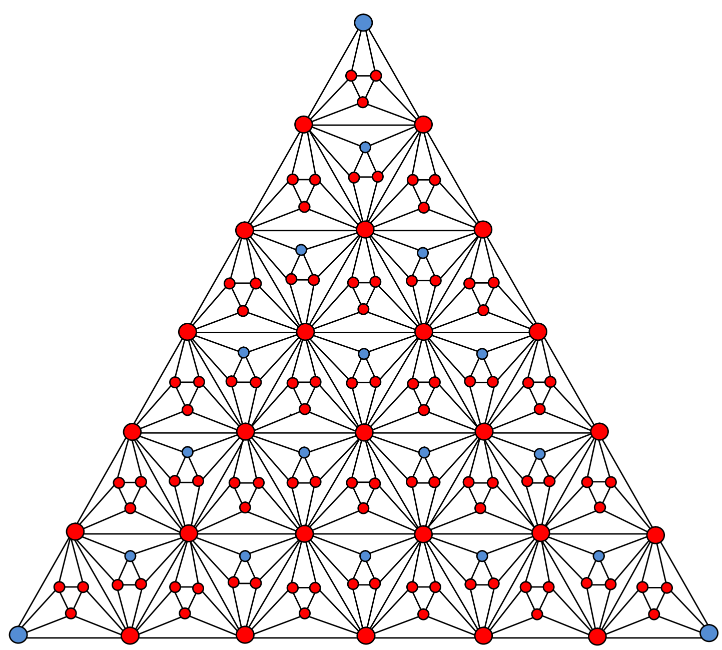


Figure 1: Type 3 triangular Hex-derived network of dimension 7 ($THDN3[7]$)

Methodology and Future Directions

In 2019, Wei et. al. [37] have enumerated several degree-based topological indices of the $HDN3[n]$ and $THDN3[n]$ networks. These indices are essentially computed separately as well as directly by using their respective degree-based formulas which are mentioned in Table 1. Instead, one can generate a closed-form of M-polynomial for the concerned networks. In this paper, we build a M-polynomial for the $THDN3[n]$ network in Section 2 and thereafter derive the related standard degree-based topological indices from the M-polynomial. In addition, we graphically represent the M-polynomial and associated degree-based topological indices of the $THDN3[n]$ network in Sections 3, and conclude in Section 4. One can look for a similar investigation in future for some subdivided graphs mentioned in [25, 34].

2 Deriving M-polynomial of *THDN*3[*n*] Network

The following theorem deals with a closed-form of M-polynomial of the third type of triangular Hex-derived network of dimension n (≥ 4).

Theorem 2.1 *Let us consider the third type of triangular Hex-derived network of dimension n (≥ 4) which is denoted as $THDN3[n]$. Then the M-polynomial of the $THDN3[n]$ network is given by*

$$\begin{aligned} M(THDN3[n]; x, y) &= (3n^2 - 6n + 9)x^4y^4 + (18n - 30)x^4y^{10} \\ &\quad + (6n^2 - 30n + 36)x^4y^{18} + (3n - 6)x^{10}y^{10} \\ &\quad + (6n - 18)x^{10}y^{18} + \frac{1}{2}(3n^2 - 21n + 36)x^{18}y^{18} \end{aligned}$$

Proof: We first itemize the cardinalities of the vertex and edge sets of the $THDN3[n]$ network by observing Figure 1 of $THDN3[7]$ network and they are given by $|V(THDN3[n])| = \frac{1}{2}(7n^2 - 11n + 6)$, and $|E(THDN3[n])| = \frac{1}{2}(21n^2 - 39n + 18)$. Now we partition the vertex set V of $THDN3[n]$ network into three non-empty subsets based on the different degrees of its vertices. They are

$$\begin{aligned} V_1(THDN3[n]) &= \{u \in V(THDN3[n]) : d(u) = 4\}, \\ V_2(THDN3[n]) &= \{u \in V(THDN3[n]) : d(u) = 10\}, \\ V_3(THDN3[n]) &= \{u \in V(THDN3[n]) : d(u) = 18\}, \end{aligned}$$

and the cardinality of each of the above vertex sets is

$$|V_1(THDN3[n])| = 3(n^2 - 2n + 2),$$
$$|V_2(THDN3[n])| = 3(n - 2),$$
$$|V_3(THDN3[n])| = \frac{1}{2}(n - 2)(n - 3).$$

In addition, based on the degrees of the end vertices of each edge, we divide the edge set E of $THDN3[n]$ network into six non-empty disjoint sub-classes, as

$$E_1(THDN3[n]) = E_{\{4,4\}} = \{e = uv \in E(THDN3[n]) : d(u) = 4, d(v) = 4\},$$
$$E_2(THDN3[n]) = E_{\{4,10\}} = \{e = uv \in E(THDN3[n]) : d(u) = 4, d(v) = 10\},$$
$$E_3(THDN3[n]) = E_{\{4,18\}} = \{e = uv \in E(THDN3[n]) : d(u) = 4, d(v) = 18\},$$
$$E_4(THDN3[n]) = E_{\{10,10\}} = \{e = uv \in E(THDN3[n]) : d(u) = 10, d(v) = 10\},$$
$$E_5(THDN3[n]) = E_{\{10,18\}} = \{e = uv \in E(THDN3[n]) : d(u) = 10, d(v) = 18\},$$
$$E_6(THDN3[n]) = E_{\{18,18\}} = \{e = uv \in E(THDN3[n]) : d(u) = 18, d(v) = 18\},$$

and their cardinalities are given by

$$|E_1(THDN3[n])| = 3n^2 - 6n + 9,$$
$$|E_2(THDN3[n])| = 18n - 30,$$
$$|E_3(THDN3[n])| = 6n^2 - 30n + 36,$$
$$|E_4(THDN3[n])| = 3n - 6,$$
$$|E_5(THDN3[n])| = 6n - 18,$$
$$|E_6(THDN3[n])| = \frac{1}{2}(3n^2 - 21n + 36).$$

Thus by definition, the M-polynomial of $THDN3[n]$ network is

$$\begin{aligned}
&M(THDN3[n]; x, y)\\
&= \sum_{i \le j} m_{i,j} x^i y^j, \quad \text{where} \quad i, j \in \{4, 10, 18\}\\
&= \sum_{uv \in E_1(THDN3[n])} m_{4,4} x^4 y^4 + \sum_{uv \in E_2(THDN3[n])} m_{4,10} x^4 y^{10}\\
&\quad + \sum_{uv \in E_3(THDN3[n])} m_{4,18} x^4 y^{18} + \sum_{uv \in E_4(THDN3[n])} m_{10,10} x^{10} y^{10}\\
&\quad + \sum_{uv \in E_5(THDN3[n])} m_{10,18} x^{10} y^{18} + \sum_{uv \in E_6(THDN3[n])} m_{18,18} x^{18} y^{18}
\end{aligned}$$

$$= |E_1(THDN3[n])|\ x^4y^4 + |E_2(THDN3[n])|\ x^4y^{10}$$
$$+ |E_3(THDN3[n])|\ x^4y^{18} + |E_4(THDN3[n])|\ x^{10}y^{10}$$
$$+ |E_5(THDN3[n])|\ x^{10}y^{18} + |E_6(THDN3[n])|\ x^{18}y^{18}$$
$$= (3n^2 - 6n + 9)x^4y^4 + (18n - 30)x^4y^{10} + (6n^2 - 30n + 36)x^4y^{18}$$
$$+ (3n - 6)x^{10}y^{10} + (6n - 18)x^{10}y^{18} + \frac{1}{2}(3n^2 - 21n + 36)x^{18}y^{18}. \ \square$$

In the following Theorem 2.2, we derive the associated standard degree-based topological indices of a given network $G = THDN3[n]$ from the above M-polynomial $M(G; x, y)$ (proposed in Theorem 2.1), by using the derivation formulas in terms of differentiation or integration (or both) over the M-polynomial which are reported in Table 1 [9].

Theorem 2.2 *Let $THDN3[n]$ be the third type of triangular Hex-derived network of dimension n (≥ 4). Then*

1. $M_1(THDN3[n]) = 6(35n^2 - 101n + 78)$.
2. $M_2(THDN3[n]) = 6(161n^2 - 593n + 588)$.
3. ${}^mM_2(THDN3[n]) = \frac{119}{432}n^2 - \frac{839}{2700}n + \frac{749}{3600}$.
4. $R_\alpha(THDN3[n]) = 4^{2\alpha}(3n^2 - 6n + 9) + 40^\alpha(18n - 30) + 72^\alpha(6n^2 - 30n + 36) + 10^{2\alpha}(3n - 6) + 180^\alpha(6n - 18) + \frac{18^{2\alpha}}{2}(3n^2 - 21n + 36)$.
5. $RR_\alpha(THDN3[n]) = \frac{1}{4^{2\alpha}}(3n^2 - 6n + 9) + \frac{1}{40^\alpha}(18n - 30) + \frac{1}{72^\alpha}(6n^2 - 30n + 36) + \frac{1}{10^{2\alpha}}(3n - 6) + \frac{1}{180^\alpha}(6n - 18) + \frac{1}{2 \times 18^{2\alpha}}(3n^2 - 21n + 36)$.
6. $SDD(THDN3[n]) = \frac{112}{3}n^2 - \frac{307}{3}n + \frac{413}{5}$.
7. $H(THDN3[n]) = \frac{91}{66}n^2 - \frac{997}{660}n + \frac{541}{1540}$.
8. $ISI(THDN3[n]) = \frac{861}{22}n^2 - \frac{2193}{22}n + \frac{5112}{77}$.
9. $AZI(THDN3[n]) = \frac{4^6}{6^3}(3n^2 - 6n + 9) + \frac{40^3}{12^3}(18n - 30) + \frac{72^3}{20^3}(6n^2 - 30n + 36) + \frac{10^6}{18^3}(3n - 6) + \frac{180^3}{26^3}(6n - 18) + \frac{18^6}{2 \times 34^3}(3n^2 - 21n + 36)$

$$= \frac{9036126764}{5527125}n^2 - \frac{487061056970516}{65572705575}n + \frac{3053788898907296}{327863527875}.$$

Proof: The aforementioned topological indices for the $THDN3[n]$ network can be obtained directly by applying the relevant derivation formulas (as listed in the sixth column of Table 1) to our proposed M-polynomial $M(THDN3[n];x,y)$. The proof of this theorem is left as an exercise to the curious reader. □

Table 1: Derivation formulas to find the degree-based topological indices from the M-polynomial of a graph G.

Sl. No.	Topological Index	Notation	Formula of Topological Indices	f(x,y)	Derivation from $(M(G;x,y))$
1.	First Zagreb Index [14]	$M_1(G)$	$\sum_{uv\in E(G)} (d(u)+d(v))$	$x+y$	$(D_x+D_y)(M(G;x,y))\|_{x=y=1}$
2.	Second Zagreb Index [14]	$M_2(G)$	$\sum_{uv\in E(G)} (d(u)d(v))$	xy	$(D_xD_y)(M(G;x,y))\|_{x=y=1}$
3.	Modified Second Zagreb Index [23]	${}^mM_2(G)$	$\sum_{uv\in E(G)} \frac{1}{d(u)d(v)}$	$\frac{1}{xy}$	$(S_xS_y)(M(G;x,y))\|_{x=y=1}$
4.	General Randić Index [5]	$R_\alpha(G)$	$\sum_{uv\in E(G)} (d(u)d(v))^\alpha$	$(xy)^\alpha$	$(D_x^\alpha D_y^\alpha)(M(G;x,y))\|_{x=y=1}$
5.	Inverse Randić Index [2]	$RR_\alpha(G)$	$\sum_{uv\in E(G)} \frac{1}{(d(u)d(v))^\alpha}$	$\frac{1}{(xy)^\alpha}$	$(S_x^\alpha S_y^\alpha)(M(G;x,y))\|_{x=y=1}$
6.	Symmetric Division (Deg) Index [36]	$SDD(G)$	$\sum_{uv\in E(G)} \left\{\begin{matrix}\frac{\min(d(u),d(v))}{\max(d(u),d(v))}\\ +\frac{\max(d(u),d(v))}{\min(d(u),d(v))}\end{matrix}\right\}$	$\frac{x^2+y^2}{xy}$	$(D_xS_y+D_yS_x)(M(G;x,y))\|_{x=y=1}$
7.	Harmonic Index [11]	$H(G)$	$\sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}$	$\frac{2}{x+y}$	$2S_xJ(M(G;x,y))\|_{x=1}$
8.	Inverse Sum (Indeg) Index [36]	$ISI(G)$	$\sum_{uv\in E(G)} \frac{d(u)d(v)}{d(u)+d(v)}$	$\frac{xy}{x+y}$	$S_xJD_xD_y(M(G;x,y))\|_{x=1}$
9.	Augmented Zagreb Index [12]	$AZI(G)$	$\sum_{uv\in E(G)} \left\{\frac{d(u)d(v)}{d(u)+d(v)-2}\right\}^3$	$\left(\frac{xy}{x+y-2}\right)^3$	$S_x^3Q_{-2}JD_x^3D_y^3(M(G;x,y))\|_{x=1}$

In the Table 1, $D_x(f(x,y)) = x\frac{\partial(f(x,y))}{\partial x}$, $D_y(f(x,y)) = y\frac{\partial(f(x,y))}{\partial y}$,

$$S_x(f(x,y)) = \int_0^x \frac{f(t,y)}{t}dt, \quad S_y(f(x,y)) = \int_0^y \frac{f(x,t)}{t}dt,$$

$$J(f(x,y)) = f(x,x), \qquad Q_\alpha(f(x,y)) = x^\alpha f(x,y), \ \alpha \neq 0.$$

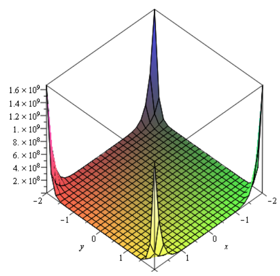
3 Graphical View of the M-polynomial of $THDN3[n]$ and Related Data

Table 2 represents the M-polynomials and the corresponding degree-based topological indices of the $THDN3[n]$ network for various dimensional values of n ($4 \leq n \leq 8$) by using Theorems 2.1 and 2.2. The numerical value of dimension n can be further increased as needed. From the table, it is clear that the value of topological indices of the network is increasing when the value of the dimension is increasing.

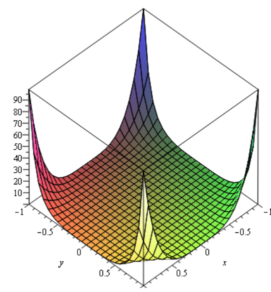
Table 2: M-polynomials and degree-based topological indices of the $THDN3[n]$ network at different values of n.

Sl.	Dimension	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
	M-polynomial / Topological Index	$33x^4y^4 + 42x^4y^{10} + 12x^4y^{18} + 6x^{10}y^{10} + 6x^{10}y^{18}$	$54x^4y^4 + 60x^4y^{10} + 36x^4y^{18} + 9x^{10}y^{10} + 12x^{10}y^{18} + 3x^{18}y^{18}$	$81x^4y^4 + 78x^4y^{10} + 72x^4y^{18} + 12x^{10}y^{10} + 18x^{10}y^{18} + 9x^{18}y^{18}$	$114x^4y^4 + 96x^4y^{10} + 120x^4y^{18} + 15x^{10}y^{10} + 24x^{10}y^{18} + 18x^{18}y^{18}$	$153x^4y^4 + 114x^4y^{10} + 180x^4y^{18} + 18x^{10}y^{10} + 30x^{10}y^{18} + 30x^{18}y^{18}$
1	First Zagreb Index	1404	2688	4392	6516	9060
2	Second Zagreb Index	4752	9888	16956	25956	36888
3	Modified Second Zagreb Index	3.3725	5.5409	8.2603	11.5306	15.3518
4	General Randić Index ($\alpha = 1/2$)	639.9532	1205.9403	1951.7509	2877.3849	3982.8422
5	Inverse Randić Index ($\alpha = 1/2$)	17.3522	29.1906	44.1098	62.1099	83.1909
6	Symmetric Division (degree) Index	270.6	504.2667	812.6	1195.6	1653.2667
7	Harmonic Index	16.3695	27.2680	40.9240	57.3377	76.5089
8	Inverse Sum (indeg) Index	293.8442	546.3896	877.2078	1286.2987	1773.6623
9	Augmented Zagreb Index	5760.9086	13046.9302	23602.6903	37428.1890	54523.4261

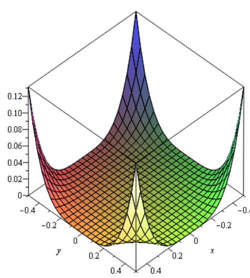
We use Maple™ 13 software to plot the surfaces of the M-polynomial (see Figure 2) of the triangular Hex-derived network of type 3 of dimension 7 ($THDN3[7]$) in different ranges of x and y. The domains of x and y are $-2 \leq x, y \leq 2$ in Figure 2(a), $-1 \leq x, y \leq 1$ in Figure 2(b), and $-0.5 \leq x, y \leq 0.5$ in Figure 2(c).



(a) The plot of the M-polynomial of $THDN3[7]$, where $-2 \leq x, y \leq 2$.



(b) The plot of the M-polynomial of $THDN3[7]$, where $-1 \leq x, y \leq 1$.



(c) The plot of the M-polynomial of $THDN3[7]$, where $-0.5 \leq x, y \leq 0.5$.

Figure 2: The plot of the M-polynomial of $THDN3[7]$ in different regions of x and y.

Furthermore, by looking over the wide range of values (in Table 2) of the degree-based topological indices of $THDN3[n]$ for different values of n, we plot the curves for the values of the first Zagreb, second Zagreb, general Randić ($\alpha = 1/2$) and augmented Zagreb indices in Figure 3, the values of the modified second Zagreb, inverse Randić ($\alpha = 1/2$) and harmonic indices in Figure 4, and the values of the symmetric division (deg) and inverse sum (indeg) indices in Figure 5.

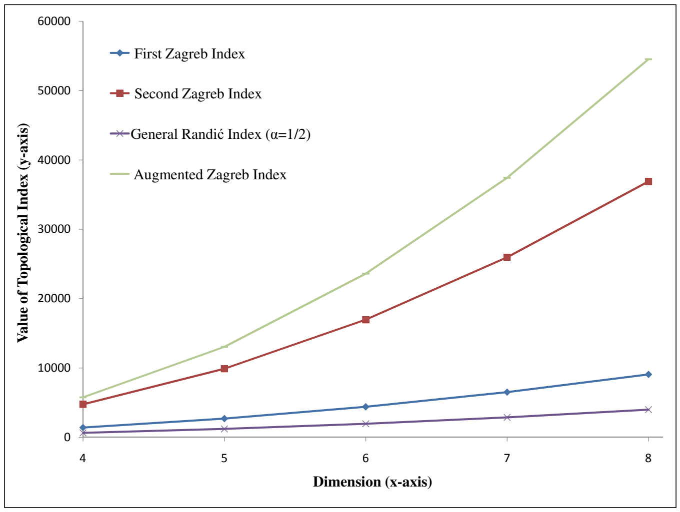


Figure 3: A plot of first Zagreb, second Zagreb, general Randić ($\alpha = 1/2$), and augmented Zagreb indices of $THDN3[n]$ for different values of n ($4 \leq n \leq 8$).

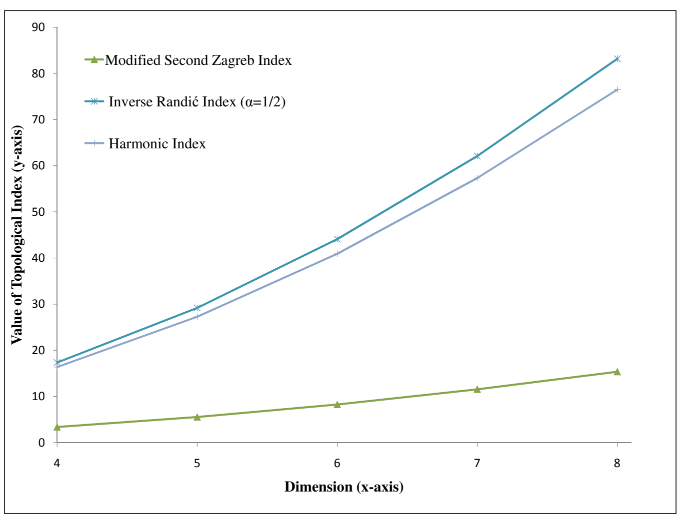


Figure 4: A plot of modified second Zagreb, inverse Randić ($\alpha = 1/2$) and harmonic indices of $THDN3[n]$ for different values of n ($4 \leq n \leq 8$).

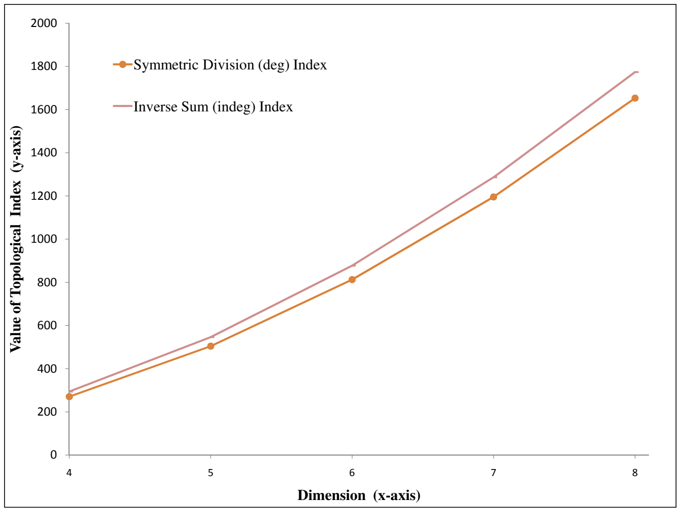


Figure 5: A plot of symmetric division (deg) and inverse sum (indeg) indices of $THDN3[n]$ for different values of n $(4 \leq n \leq 8)$.

4 Conclusion

In this paper, we have adopted the third type of triangular Hex-derived network of dimension n which has diverse applications in pharmaceutical sciences, electronics, and communication networks. For the $THDN3[n]$ network, instead of computing the standard degree-based topological indices, we have evaluated a general and closed-form of M-polynomials and therefore derived the various degree-based topological indices directly. Also, we have pictorially represented the nature of the respective M-polynomials, and the nine associated topological indices for different dimensions. It shows that all these indices have the same behaviour.

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