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Small Longest Tandem Scattered Subsequences

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Abstract

We consider the problem of identifying tandem scattered subsequences within a string. Our algorithm identifies a longest subsequence which occurs twice without overlap in a string. This algorithm is based on the Hunt-Szymanski algorithm, therefore its performance improves if the string is not self similar, which occurs naturally on strings over large alphabets. Our algorithm relies on new results for data structures that support dynamic longest increasing sub-sequences. In the process we also obtain improved algorithms for the decremental string comparison problem.

Keywords: Hunt-Szymanski algorithm, longest increasing subsequence, Tandem, sub-sequence

1 Introduction

In this paper we study longest common scattered sub-sequences (LCSS). Given two strings P and S the LCSS is used extensively as a measure of similarity. In particular we consider a variant of this problem, where the LCSS must occur twice without overlap in an initial string F. We study

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algorithms and data structures that are relevant for this goal. Namely we use the Hunt-Szymanski algorithm [1977] and present new results for data structures that maintain information about the longest increasing subsequence of a dynamic sequence of numbers and new algorithms for the decremental string comparison problem. Specifically we get the next results:

1. A data structure to maintain the longest increasing subsequence (LIS) of a dynamic list of numbers. This structure can be used to efficiently: `Append` a number at the end of the list; remove the current minimum value from the sequence (`ExtractMin`); obtain a current longest increasing subsequence (`GetLIS`). When the list contains $\ell \geq 2$ elements, the operation `Append` requires $O(\log \ell)$ time[3]. If the size of the LIS is $\lambda \geq 2$ then the `ExtractMin` operation requires $O(\lambda \log \ell)$ time. See Theorem 1. We further improve these bounds by analyzing batches of operations and assuming the final sequence is empty. In this case `ExtractMin` requires $O(\lambda(1 + \log(\min\{\lambda, \ell/\lambda\})))$ amortized time per operation and `Append` requires $O(1)$ amortized time, when the numbers are inserted in decreasing sequences of size at least λ elements. See Theorem 2. This structure uses optimal $O(\ell)$ space.

2. Using the Hunt and Szymanski [9] reduction from LCSS to LIS we obtain new bounds for the decremental string comparison problem. In particular, for a given string F of size $n > 1$, we show that it is possible to obtain all the LCSS values for all the pairs of strings P and S such that $F = P.S$ in $O(\min\{n, \ell\}\lambda(1 + \log(\min\{\lambda, \ell/\lambda\})) + n\lambda + \ell)$ time, where $\lambda \geq 2$ is the size of the LCSS and $\ell \geq 2$ is the number of pairs of positions in F that contain the same letter. Therefore it is possible to determine the LTSS within this time, i.e., the LCSS which occurs twice without overlap in a string F.

2 The Problem

Let us start by describing the longest tandem scattered sub-sequence (LTSS) problem of a given string F. We will use a running example with $F =$ `AGCGAACGGGTA`. The meaning of tandem is that the sub-sequence needs to occur twice without overlap in F. Therefore F can be partitioned into a

[3]Note that to simplify expressions as $O(1+\log \ell)$ we impose restrictions on parameters such as $\ell \geq 2$. This also avoids invalid expressions such as when $\ell = 0$. In general the complexity of the excluded cases is $O(1)$.

prefix P and a suffix S, i.e., $F = P.S$, such that the desired scattered sub-sequence is a longest common scattered sub-sequence (LCSS) between P and S. To determine which partition yields the overall largest sub-sequence it is necessary to test all such partitions.

Let us consider the partition with $P =$ `AGCG` and $S =$ `AACGGGTA`. The LCSS is the longest string that occurs as a scattered sub-sequence of P and S. Figure 1 illustrates that the string `ACG` is a longest common scattered sub-sequence of P and S. A common sub-sequence can be defined as a set of pairs (i, j) where i is an index over P and j an index over S and the i-th letter of P is equal to the j-th letter of S, represented as $P(i) = S(j)$. All numbers i must be distinct among themselves and all numbers j must be distinct among themselves. Moreover sorting the pairs by i must also yield a sorted sequence by j. In our example the LCSS for P and S can be represented by the set $\{(1, 1), (3, 3), (4, 4)\}$.

Figure 1 also shows an LCSS for a second prefix suffix decomposition of $F = P'.S'$. This second decomposition is related to the first as $P' = P.\mathtt{A}$ and in fact the LCSS is similar to the previous LCSS, with the extra character `A`, i.e., `ACGA`.

In this example the LCSS between P' and S' is the desired overall LTSS.

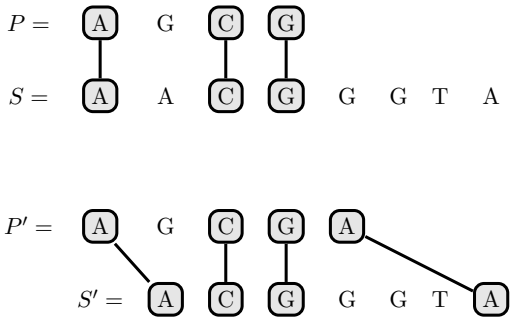

Figure 1: Example of LCSS for two prefix suffix decompositions of F, $F = P.S$ and $F = P'.S'$

3 Decremental Comparison and Hunt-Szymanski

In this section we present the main ideas of an algorithm that computes LTSS. Given a string F we can reduce this problem to computing the size of the LCSSs for all prefix and suffix decompositions of F, i.e., for all $P.S = F$, where P is a prefix and S is a suffix. The resulting tandem can be obtained

from the overall largest LCSS. The pseudo code for this process is shown in Algorithm 1, where $F(i')$ represents the i'-th letter of F, i.e., the letters indexes start at 1.

This process involves n LCSS computations, when the size of F is n. This computation is referred in line 8 of Algorithm 1. Each LCSS can be determined with the classical dynamic programming table between P and S. Table D is a bi-dimensional array that stores integers. Each value $D[i, j]$ represents the size of the LCSS between the prefix of P with i letters and the prefix of S with j letters. The coordinate i ranges from 0 to the size of P, likewise coordinate j ranges between 0 and the size of S. The value 0 represents the empty prefix.

The values $D[i, j]$ can be computed locally according to the equalities bellow, where $P(i)$ denotes the i-th letter of P and $S(j)$ the j-th letter of S:

$$D[i,j] = 0 \qquad \text{if } i = 0 \text{ or } j = 0 \tag{1}$$

$$D[i,j] = D[i-1,j-1] + 1 \qquad \text{if } i,j > 0 \text{ and } P(i) = S(j) \tag{2}$$

$$D[i,j] = \max\{D[i,j-1], D[i-1,j]\} \qquad \text{if } i,j > 0 \text{ and } P(i) \neq S(j) \tag{3}$$

Let us consider a running example with $P = \texttt{AGCG}$ and $S = \texttt{AACGGGTA}$. The values of table $D[i, j]$ are shown in the top portion of Figure 2. For example the value $D[4, 3]$ is 2, which means that the LCSS between `AGCG` and `AAC` has size 2. This table requires $O(n^2)$ time to build, when P and S have $O(n)$ size. In the example of Figure 2 the desired maximum value is $D[4, 8] = 3$, it is this value that is obtained by the `GetMaxD` operation in Algorithm 1.

The LCSSs can be recovered with tracebacks. A traceback is a pointer from a cell $D[i, j]$ to one of its neighboring cells $D[i - 1, j]$, $D[i, j - 1]$ or $D[i-1, j-1]$. The resulting paths represent the corresponding LCSSs. The diagonal tracebacks represent matches between the corresponding strings, we show only these tracebacks in Figure 2. In our example there is a diagonal traceback from $D[4, 4]$, representing the fact that both strings end with the letter `G`. Let us then consider $S' = \texttt{ACGGGTA}$ and $P' = P.\texttt{A}$. We also need to compute the D table for these strings, shown at in the middle of Figure 2. To avoid confusion we refer to this table as D' in the text. In Algorithm 1 all D tables are always represented by D.

We aim to compute a representation of table D' in $O(n)$ time, instead of $O(n^2)$, i.e., we aim to reduce the time bound of the `UpdateD` operation. First let us highlight the changes between D and D'. Column $D[i, 1]$ is removed, which corresponds to removing `A` from S. Row $D'[5, j]$ is inserted, which corresponds to appending `A` to P. Several values are maintained,

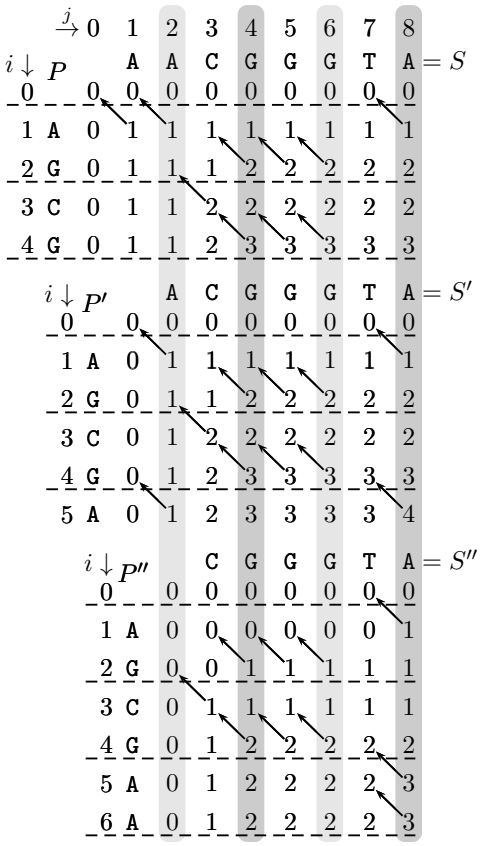

Figure 2: Illustration of tables D, D' and D'' with diagonal tracebacks.

$D[i, j+1] = D'[i, j]$. The remaining values decrease by 1, i.e., $D'[i, j] = D[i, j+1] - 1$. In our example only the values of column $D'[i, 0]$ decrease, the remaining values are maintained. To determine which cells change and which remain constant we consider another representation of table D. The representation used in the Hunt-Szymanski, Algorithm [1977].

Let us now focus on how to efficiently compute decremental string comparison, i.e., a simple and efficient way to obtain table D' from table D. Which is the process we will use in line 8 of Algorithm 1. Let us start by reviewing and augmenting the Hunt-Szymanski algorithm [9]. The algorithm works by reducing the LCSS to the problem of determining a longest increasing sub-sequence of numbers (LIS). This reduction is illustrated in Figure 3. It works in two steps. In the first step it processes S. For every letter b

Algorithm 1 `SizeOfLTSS`(F)

```
1: n ← |F|
2: P ← ε                                          ▷ Empty string.
3: S ← F
4: x ← 0                                          ▷ Initial value for size of LTSS.
5: for i' = 1, ..., n − 1 do
6:     P ← P.F(i')                                ▷ Appends letter to P.
7:     Pop(S)                                     ▷ Removes the first letter of S.
8:     UpdateD(F(i'))   ▷ Update D according to the change in P and S.
9:     λ ← GetMaxD()    ▷ λ becomes the size of the corresponding LCSS.
10:    if x < λ then
11:        x ← λ                                  ▷ Found a new maximum.
12:    end if
13: end for
14: return x                                      ▷ Size of the overall LTSS.
```

in S it computes the list of positions where b occurs in S, represented by $M_S(b)$. In the second step it processes P, from left to right, and produces a list of numbers P_S. For every letter b of P the list $M_S(b)$ is appended to the current list of numbers.

The resulting list P_S consists of a list of positions of S, where the same position may appear several times. Hence selecting a subsequence from P_S is equivalent to choosing letters from S and P simultaneously. In our example a resulting longest increasing subsequence is $1 < 3 < 4$. Selecting these letters from S yields the desired common subsequence `ACG`. To avoid selecting the same letter from S repeatedly the LIS needs to be strictly increasing. Moreover, to guarantee that a letter from P is selected only once the lists $M_S(b)$ are sorted in decreasing order and this order is used to build P_S.

The Hunt-Szymanski algorithm then proceeds to efficiently compute the LIS. In this context we represent the list of numbers by L, abstracting away the process that was used to produce it, i.e. $L = P_S$. We will use ℓ to refer to the size of L. To determine the LIS the algorithm uses a sequence of threshold lists T_k. List T_k contains the element i of L if the longest increasing subsequence which ends with i, of the first elements of L up i, has size k. The top box of Figure 4 shows this threshold structure for the sequence L we are considering. In this example we can observe that T_2 contains the value 6, which corresponds to the first 6 in L. This occurs because the LIS up to that element has size 2, namely it could be $2, 6$. In

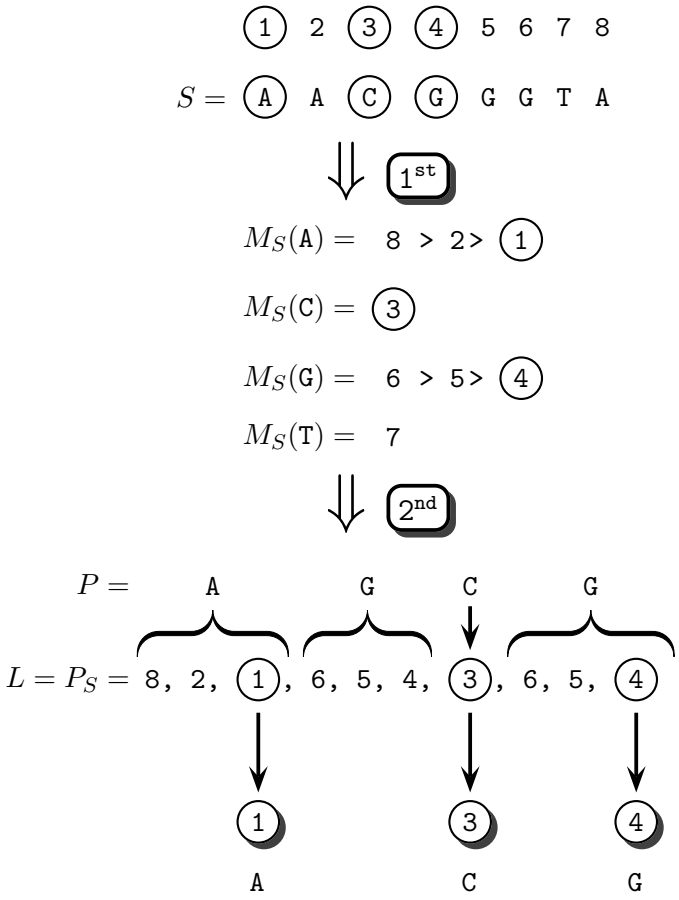

Figure 3: The Approach

another example we can observe that T_3 also has a value 6, this corresponds to the second 6 in L and is in this list because the LIS up to this element has size 3, namely it could be $2, 4, 6$.

Now let us return to the decremental string problem and study how this data structure is affected when P changes to P' and S changes to S'. The top part of Figure 2 shows the dynamic programming table D, for P and S. The figure also illustrates D' and D'' for the consecutive decompositions that append letters to P and remove letters from the beginning of S. This figure serves to illustrate the relation between the D table and the T_k lists. Figure 2 shows only diagonal tracebacks, as these are the only ones that appear in

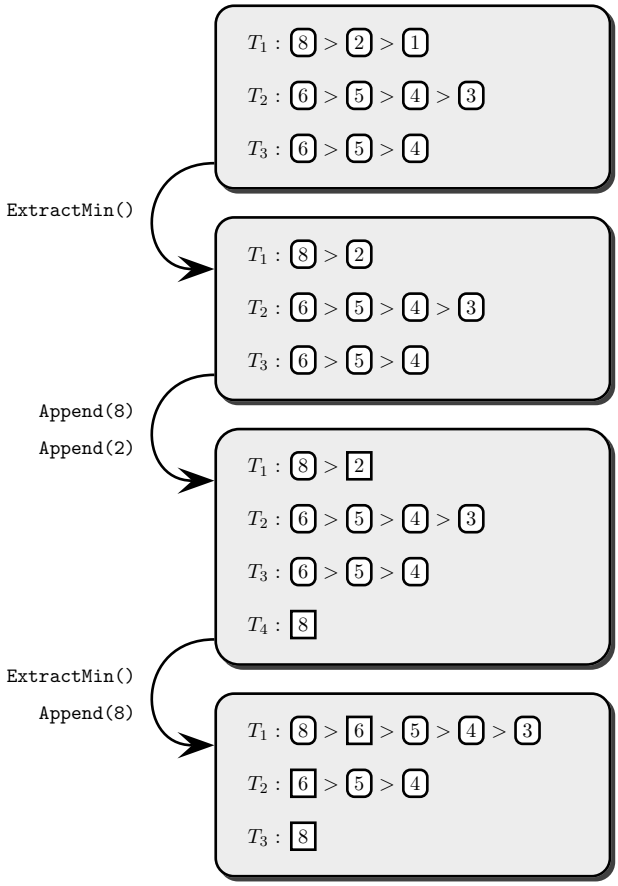

Figure 4: Dynamic LIS computation. The top box shows the T_k lists for P and S.

the T_k lists. For example, if we consider the cells in D that are equal to 2, the list T_2 gives a representation of this set. The cells $(2,6),(2,5),(2,4)$ and $(3,3)$ are the respective diagonal tracebacks. The T_k lists store only the j coordinates, therefore the list T_2 contains $6 < 5 < 4 < 3$. In general the list T_k stores the decreasing j coordinates of the cells with D value k and diagonal tracebacks, i.e., there is an i such that $D[i,j] = k$ and $P(i) = S(j)$.

Computing the threshold structure is done incrementally by processing the elements of L from left to right. Therefore the original Hunt-Szymanski algorithm already supports the `Append` operation, which updates the structure when a new number is appended. For completeness and latter analysis we present this process in Algorithm 3. Each list T_k is stored in decreas-

ing order. With this organization the sequence of tail elements is kept in increasing order throughout the execution of the algorithm. After processing our sample list L the resulting sequence of tail elements is $1 < 3 < 4$. This order is used to determine in which T_k a given element of L should be inserted. Let us consider our running example and start with all the T_k lists empty. The first 8 initializes T_1. The 2 is also appended to T_1 because $8 > 2$, likewise 1 is also appended to T_1 because $2 > 1$. Number 6 initializes list T_2, because $1 < 6$, which becomes the current sequence of tail elements. Numbers $5, 4$ and 3 are also appended to T_2, as they are all greater than 1 and in decreasing order. The number 6 initializes list T_3, because $1 < 3 < 6$. Likewise 5 and 4 are also appended to T_3. Since list T_3 is not empty, we know that L contains an increasing subsequence of length 3. Since T_4 was left empty, there is no increasing subsequence of length 4. Therefore the size of a LIS in our example is 3.

4 Efficient Dynamic LIS

We now focus on adapting the threshold structure to obtain the longest subsequence which occurs twice without overlap, in a string F. Consider $F =$ `AGCGAACGGGTA`, in which case the subsequence could be `ACGA`, which has size 4. To obtain this value we execute Algorithm 1 using the threshold structure to implement line 8, meaning that we aim to update D to reflect the changes in P and S. The procedure we propose to perform this process is sketched in Algorithm 2. It requires two primitives from the dynamic LIS data structure. The `ExtractMin` operation removes the minimum element of L and the `Append`(i) operation appends the number i to the end of L. The rational for these operations is that `Append` is used to update D when a letter is added to P and removing the letter from S implies removing the minimum position, from L and from $M_S(c)$, were c is the letter being removed, i.e., $c = F(i')$. Note that this latter operation can be done in $O(1)$ time because in $M_S(c)$ the minimum occurs at the last position of the list. Appending the letter c to P corresponds to increasing the list L by appending the new $M_S(c)$ positions to L, this is performed by the **for** cycle in Algorithm 2.

Hence the crucial component to obtain an efficient implementation of Algorithm 1 is a data structure that can compute the `Append` and `ExtractMin` operations quickly. Assume that we have the T_k lists for the strings P and S. We aim to update this structure for strings $P' = P.\mathtt{A}$ and

Algorithm 2 `UpdateD(`c`)`

```
1: if 0 < ℓ then                          ▷ No extraction if L is empty.
2:     ExtractMin()
3: end if
4: Trim(M_S(c))                           ▷ Remove the min also from M_S(c).
5: for all i ∈ M_S(c) do
6:     Append(i)
7: end for
```

$S' =$ `ACGGGTA`, i.e., we want to append a letter to P and remove the first letter from S. Removing the first letter from S changes the $M_S(b)$ lists, in particular all the positions are offset by 1, for example $M_{S'}(\texttt{G}) = \texttt{5}, \texttt{4}, \texttt{3}$. This offset does not alter the relative order of the numbers nor the shape of the threshold structure. Therefore we ignore this offset, to simplify the exposition, and also to reduce the complexity of the algorithm. Instead assume that we start numbering the positions of S' at 2. Now the only change is to $M_{S'}(\texttt{A}) = \texttt{8}, \texttt{2}$, which looses position `1`, as it is no longer part of S'. In general the position that gets removed is the overall minimum. Hence we need to apply an `ExtractMin` operation to the threshold structure. This operation should remove all the instances of the minimum in L, in this case all the instances of 1. In this particular example there is only one instance, but in general there can be several such occurrences. Algorithm 4 shows the precise pseudo-code for this operation, let us now illustrate it with our running example.

Due to the decreasing order of the T_k lists and increasing order of the tail elements it is straightforward to locate the overall minimum element. The minimum is always the tail element of T_1. Notice that even if there are several instances of the minimum in L there is only one element in T_1, because of our approach of discarding duplicated elements. Now remove this element from T_1. The resulting structure still maintains the necessary orders as the sequence of the tail elements becomes $\texttt{2} < \texttt{3} < \texttt{4}$, and the internal order of the T_k's was not altered. This structure is shown in the second box of Figure 4. This is indeed the same structure that can be obtained from the sequence $\texttt{8}, \texttt{2}, \texttt{6}, \texttt{5}, \texttt{4}, \texttt{3}, \texttt{6}, \texttt{5}, \texttt{4}$. Thus, in this case, no further work is required.

Since P' contains an extra `A`, we need to append the list $M_{S'}(\texttt{A})$ to L. Therefore we execute the operations `Append(8)` and `Append(2)`. This alters the T_k lists, as explained above. Appending the number `8`, initializes T_4,

because $\texttt{2} < \texttt{3} < \texttt{4} < \texttt{8}$. Appending the number 2 does not produce any change because it is already the tail of T_1 and we do not store repetitions in the T_k lists. In this case we simply drop the element, Section 4.1 describes more a elaborated process that is used when the size of the LIS is not enough and we want to retrieve an actual such sequence. At this point we obtained a LIS of size 4 which identifies a LCSS of P' and S' that is our goal subsequence of F. However to make sure this is indeed the longest subsequence we must continue Algorithm 1 and update the T_k lists for $P'' = P'.\texttt{A}$ and $S'' = \texttt{CGGGTA}$. Again we begin by computing `ExtractMin`. Notice that, because we do not store repetitions, this procedure removes both instances of the number 2. This time the operation is more elaborated because after removing the 2 from T_1 the resulting sequence of tail elements is no longer increasing. Note that 8 is the tail of T_1 and 3 is the tail of T_2 and $8 > 3$. To solve this problem we could transfer the 3 from T_2 to T_1, thus fixing the first inequality as $3 < 4$. However this is not correct. Note that at this point the sequence L we are considering is $\texttt{8}, \texttt{6}, \texttt{5}, \texttt{4}, \texttt{3}, \texttt{6}, \texttt{5}, \texttt{4}$, in which case T_1 should be $8 > 6 > 5 > 4 > 3$. Therefore the correct procedure is to remove all the elements from T_2 and append them to T_1. Now T_2 becomes empty so all the elements from T_3 are moved to T_2, which leads T_3 to become empty and therefore all the elements from T_4 are moved to T_3. Hence T_4 becomes empty and the process terminates because T_4 was the last list.

The general procedure for `ExtractMin` is to remove the tail element from T_1 and then transfer from T_2 to T_1 all the elements that are smaller than the current tail element. The process continues from T_{k+1} to T_k until there are no further elements to transfer, either because T_{k+1} is empty or all its elements are larger than the current tail element of T_k. This process is summarized in Algorithm 4. In Section 4.1 we formalize, extend and analyze this data structure. Our example finishes by appending `8`, which, because we do not store repetitions in our T_k lists, gets discarded and therefore does not alter the structure. For P'' and S'' the resulting LIS has size 3 and is therefore smaller than the subsequence `ACGA` obtained for P' and S'. This was in fact a desired subsequence, but the algorithm must scan the remaining pairs of prefixes and suffixes to certify this conclusion.

4.1 Implementation and Analysis

First let us discuss which data structures can be used to efficiently store the threshold data structure. In the classical Hunt-Szymanski algorithm

each T_k list can be stored in a stack, where reading the `Top` element and pushing new elements can be achieved in constant time. There is no need to pop elements from the stacks, so it is enough to store the `Top` values. These values are stored in an array so that it is possible to perform a binary search on the top elements. The procedure to execute `Append(`i`)` is to execute a binary search on the array to find k such that $\texttt{Top}(T_{k-1}) < i \leq \texttt{Top}(T_k)$. If T_k is empty assume its stack top is $+\infty$, also assume there is a sentinel list T_0 with $\texttt{Top}(T_0) = -\infty$. If for the resulting k we have $\texttt{Top}(T_k) = i$ then the procedure stops, otherwise it performs `Push(`T_k`, `i`)`.

To support the `ExtractMin` operation we prefer to use a different data structure. We represent the T_k lists using balanced binary search trees (BST), in particular red-black trees. This allows us to compute `Min(`T_k`)`, `Insert(`T_k`, `i`)`, `Remove(`T_k`, `i`)`, `Predecessor(`T_k`, `i`)`, `Split(`T_k`, `v`)` and `Concatenate(`T_k`, `v`)` in $O(\log \ell)$ time, where ℓ is the size of L. Like the Hunt-Szymanski algorithm, we keep an array `Min[`k`]` that stores the tail element of T_k, so that it can be accessed in constant time. The `Min(`T_k`)` operation finds the smallest element in T_k. When the BST of T_k is empty it returns $+\infty$. The `Insert(`T_k`, `i`)` operation inserts the number i into the BST of T_k. The `Remove(`T_k`, `i`)` operation removes the number i from the BST of T_k, if key i does not exist then an error is reported and the current process is stopped. The `Predecessor(`T_k`, `i`)` operation finds the largest element of T_k that is less than or equal to i, i.e., $\max\{j \in T_k | j \leq i\}$, the result should be a pointer to the corresponding tree node v, if no such node exists the pointer should be `NULL`. The `Split(`T_k`, `v`)` operation divides the BST of T_k in two by keeping all the nodes with keys strictly larger than v in T_k and putting v and the remaining nodes in a new BST. The operation `Concatenate(`T_k`, `v`)` joins the BST containing node v in front of the BST of T_k, assuming that all the key values in T_k are larger than or equal to the key in v and v is the maximum key value in its BST. Recall that we assume that the values in T_k are not repeated, therefore the `Insert` and `Concatenate` operations drop duplicated elements when they occur. Algorithms 3 and 4 show the pseudo-code for the `Append` and `ExtractMin` operations, respectively. Note that for the `Append` procedure the `Min[`k`]` array plays the role of the `Top` operation in the classical version.

Let us now analyze the time performance of the `Append` procedure, Algorithm 3. Without the `Min[`k`]` array the overall time would be $O((\log \ell)(\log \lambda) + \log \ell)$, where the first term accounts for the binary search in lines 4 to 10. The second term accounts for the `Insert` operation in

line 12. Using the `Min[k]` array the first term reduces to $O(\log \lambda)$ and thus the overall time becomes $O(\log \ell)$ because $\lambda \leq \ell$, since λ is the size of a subsequence of L.

Now let us analyze the `ExtractMin` procedure, Algorithm 4. The **while** loop executes at most λ times. Each execution requires $O(\log \ell)$ time for the `Predecessor`, `Split` and `Concatenate` operations. Hence we obtain a bound of at most $O(\lambda \log \ell)$ time. However an even tighter bound is possible. This operation can be bounded by $O(\sum_{k=1}^{\lambda} \log(|T_k|))$, where $|T_k|$ is the size of the list T_k. Because the log function is concave and the size of all the lists adds up to ℓ we can use Jensen's inequality [1906] to obtain an $O(1 + \lambda \log(\ell/\lambda))$ bound. The following derivation justifies the bound.

$$\begin{aligned}\sum_{k=1}^{\lambda} \log(|T_k|) &= \lambda \sum_{k=1}^{\lambda} \frac{\log(|T_k|)}{\lambda} \\ &\leq \lambda \log\left(\sum_{k=1}^{\lambda} \frac{|T_k|}{\lambda}\right) \\ &= \lambda \log(\ell/\lambda)\end{aligned}$$

This finishes the dynamic LIS contribution, which is summarized in the next Theorem.

Theorem 1 *It is possible to maintain a dynamic list with $\ell \geq 2$ numbers such that the `Append` operation can be computed in $O(\log \ell)$ time and `ExtractMin` and `GetLIS` requires $O(1 + \lambda \log(\ell/\lambda))$ time, for a longest increasing sub-sequence, of size λ.*

The `Append` and `ExtractMin` operations have just been described in this section. The `GetLIS` operation is described in A.

This result establishes some initial bounds of this data structure. However these bounds are fairly non competitive for our goals. To determine an LTSS we might generate a sequence with $\ell = O(n^2)$ elements and perform $O(\ell)$ `Append` operations and n `ExtractMin` operations. This yields an $O(n^2 \log n)$ time algorithm. Let us improve the performance of the dynamic LIS data structure. First we change the red-black BSTs to finger trees [6, 8, 7]. This means that the `Split` and `Concatenate` operations that involve the $t_i \geq 2$ tail elements of T_k require only $O(\log t_i)$ amortized time, instead of $O(\log |T_k|)$ time. Let us consider the overall algorithm, from the initial empty structure to the final one. We will analyze the overall time

Algorithm 3 `Append(`i`)`

Ensure: Updated threshold structure for L with i appended.

```
1: ℓ ← (ℓ + 1)                                  ▷ Increase size of L.
2: j ← 0
3: k ← (λ + 1)
4: while j + 1 < k do
5:     m ← ⌊(j + k)/2⌋
6:     if i > Min[m] then
7:         j ← m
8:     else
9:         k ← m
10:    end if
11: end while
12: Insert(T_k, i)
13: Min[k] ← i
14: if k = (λ + 1) then
15:    λ ← (λ + 1)                               ▷ LIS grows
16: end if
```

that is used to process a given list T_k. The following argument applies for any k but for simplicity consider that we are analyzing T_λ. We have the following inequality:

$$\sum_{i=1}^{n} t_i \leq n + (\lambda - 1)n \tag{4}$$

The left term in the inequality counts the number of elements that are moved from T_λ. The right side counts the number of elements that are removed from the data structure. The term n counts the number of elements that are actually removed from T_1, one for each `ExtractMin` operation. The term $(\lambda - 1)n$ accounts for the elements that are dropped in the middle of the data structure. In each `ExtractMin` operation at most $(\lambda - 1)$ elements are dropped, one for each T_k list, except for T_λ. Now the total time of these operations is $O(\sum_{i=1}^{n} \log t_i)$. We can obtain this value, restricted to Equation (4), by using Lagrange multipliers. We consider only one Lagrange multiplier, represented by c, because we have only one restriction. Hence the resulting Lagrangian expression is the following:

$$\sum_{i=1}^{n} \log t_i - c(\sum_{i=1}^{n} t_i - \lambda n)$$

Algorithm 4 `ExtractMin()`

Require: L is not empty
Ensure: Updated threshold structure for L without the current minimum.

```
 1: ℓ ← (ℓ − 1)                                   ▷ Decrease size of L.
 2: Remove(T_1, Min[1])
 3: k ← 2
 4: if ℓ > 0 then
 5:     while Min(T_{k−1}) ≥ Min[k] do            ▷ Assuming Min[λ + 1]= +∞
 6:         v ← Predecessor(T_k, Min(T_{k−1}))
 7:         Split(T_k, v)
 8:         Concatenate(T_{k−1}, v)
 9:         Min[k − 1] ← Min[k]
10:         k ← k + 1
11:     end while
12: end if
13: Min[k − 1] ← Min(T_{k−1})
14: if Min[λ] = +∞ then
15:     λ ← (λ − 1)                               ▷ LIS shrinks
16: end if
```

A derivative in order of t_i yields the following condition:

$$\frac{1}{t_i} = c \tag{5}$$

The derivative in order of c returns the original restriction:

$$\sum_{i=1}^{n} t_i = \lambda n \tag{6}$$

Combining both equations we obtain that $c = 1/\lambda$ and therefore $\log t_i = \log(\lambda)$. If we use the same upper bound for all the other T_k lists we obtain $O(n\lambda \log \lambda)$ total time for n `ExtractMin` operations. This yields an amortized time of $O(\lambda \log \lambda)$ per operation, provided the final structure is empty. This new bound for `ExtractMin` is not necessarily smaller than the previous $O(\lambda \log(\ell/\lambda))$, but the best of both applies.

Besides this bound for `ExtractMin` we also need a faster `Append` operation. Using the amortized performance of the finger tree data structure we obtain an $O(\log \lambda)$ amortized bound for the `Append` operation. This

Algorithm 5 `AppendBatch(`i`)`

Ensure: Updated threshold structure for L with i appended.

1: $\ell \leftarrow (\ell + 1)$ ▷ Increase size of L.
2: **if** `Min[`k`]` $< i$ **then** ▷ Value of k is maintained between calls.
3: $k \leftarrow \lambda + 1$
4: **end if**
5: **while** $k > 1$ and `Min[`T_{k-1}`]` $> i$ **do**
6: $k \leftarrow k - 1$
7: **end while**
8: `Insert(`T_k`,` i`)`
9: $\lambda \leftarrow \max(\lambda, k)$

performance can be further improved by considering how it is called from Algorithm 2. Notice that this operation is repeatedly called from the **for** loop in line 5 to append the elements in $M_S(c)$. Moreover these elements are in decreasing order of value, this fact can be explored to obtain another time bound. With the previous bound the **for** loop in line 5 would require $O(m \log \lambda)$ time, when $M_S(c)$ contained m elements. By exploring the fact that the elements in $M_S(c)$ are ordered we can obtain a bound of $O(m + \lambda)$ instead. To obtain this bound do a simple linear scan from T_λ down to the desired position, instead of a binary search. Only reset the search if necessary. A sequence of decreasing numbers is refereed to as a batch. During a batch the position k of the scan is not reset. This means that processing a batch containing m numbers requires only $O(m + \lambda)$ time. Note that this is $O(1)$ amortized time per number, when the batch contains at least λ numbers. The pseudo code for the `AppendBatch` procedure is shown in Algorithm 5, where line 8 is computed in $O(1)$ amortized time with the finger tree data structure and line 5 accumulates to $O(\lambda)$ in a decreasing sequence. Note that the local variable k preserves its value among successive calls.

We can now summarize our dynamic LIS data structure in the following theorem:

Theorem 2 *Consider the Longest Increasing Sub-Sequence of a dynamic list of numbers, which starts and finishes empty. Assuming that in total ℓ elements are inserted into the structure, in d batches of decreasing sequences and also that the* `ExtractMin` *operation is executed e times in total, then the overall time for this is bounded by $O(e\lambda(1 + \log(\min\{\lambda, \ell/\lambda\})) + \ell + d\lambda)$, where $\lambda \geq 1$ is the size of the largest overall LIS. At anytime the size of the*

current LIS can be obtained in $O(1)$ time.

We can now combine the results of Theorem 1 and 2 to obtain our bounds for the decremental string comparison problem.

Theorem 3 *Given strings P and S there exists a data structure that can be used to obtain the size of LCSS between these strings, $\lambda \geq 2$, which requires $O(\ell)$ space, where ℓ is the number of matches between P and S. This structure can be updated to the strings $P.c$ and S in $O(\lambda+|S|)$ time, where c is any letter. It can also be updated for the strings P and S', where $S = c.S'$, in $O(\lambda(1 + \log(\ell/\lambda)))$ time. A sequence of operations that starts with an empty string and inserts letters to form the string P requires $O(|P|\lambda + \ell)$ time. A sequence of operations that decrements S until it becomes empty requires $O(\min\{|S|, \ell\}\lambda(1 + \log(\min\{\lambda, \ell/\lambda\})) + |S|)$ time.*

The amortized complexities follow from Theorem 2, and the extra min that appears is a bound on the number of `ExtractMin` operations, e in Theorem 2. This number of operations is bounded simultaneously by $|S|$ and by ℓ, because we cannot remove more points than the ones that exist inside the structure. However in the case where $e < |S|$ it is necessary to add an $O(|S|)$ term. This corresponds to the case where the letter c that is being removed from S has no occurrences in P. In this case there is no call to `ExtractMin` operation but this verification still needs to be performed, which requires $O(1)$ time and must be accounted for.

Our application of computing the LTSS now follows from Theorem 3. The total amount of time the LTSS algorithm is therefore $O(\min\{n, \ell\}\lambda(1 + \log(\min\{\lambda, \ell/\lambda\})) + n + \ell)$, where $\lambda \geq 2$ is the size of the LTSS and $\ell \geq 2$ is the number of pairs of positions in F that contain the same letter.

5 Related Work

An initial efficient algorithm to compute the LTSS, for the simple case of only one string F, was given by Kosowski [15]. This algorithm required optimal $O(n^2)$ time and $O(n)$ space. Tiskin [20, Section 5.6] presented an algorithm which obtains the smallest worst case bound by exploring the Monge properties of the respective distance matrices. This property depends on the fact that the graph underlying the D table of two strings is planar. The resulting algorithm obtains the overall worst case time bound of $O(n^2(\log\log n)^2/(\log n)^2)$.

The work on incremental string comparison was initiated by Landau, Myers, and Schmidt [16], which obtained an $O(n)$ time algorithm to obtain D' from D. A simpler version, with the same performance was presented by Kim and Park [14], which is simultaneously incremental and decremental. This is the first instance of the decremental variation of the problem. This solution was presented for the edit distance. Ishida, Inenaga, Shinohara, and Takeda [11] presented an algorithm which reduced the time complexity from $O(n)$ to $O(\lambda)$ and was fully incremental. The algorithm was presented for the LCSS and they also reduced the space requirements from $O(n^2)$ to $O(n\lambda)$.

Landau, Myers, and Ziv-Ukelson [17] studied the problem of consecutive suffix alignment problem, which obtained the size of the LCSS between all the suffixes of a string A and a string B, the final version of the paper appeared in [2007]. The authors presented two algorithms for this problem, which required $O(n\lambda)$ and $O(n\lambda + n \log \sigma)$ time, where σ is the size of the alphabet of the underlying strings. Their approach uses a structure similar to the T_k lists from the Hunt-Szymanski algorithm, but contrary to our approach of Section 3 the elements are prepended to a variation of the T_k lists. Moreover their structure is not decremental. Because of these nuances the relation to LTSS is not immediate which justifies the algorithm of [15], in the same year.

A corner stone of all these results is the algorithm from Hunt and Szymanski [9], whose crucial idea was the reduction from the LCSS to the LIS, although this was not immediately clear in the original presentation. It was partially identified by [1], [2] and made explicit by [12] and independently by [19]. Interestingly the original presentation of Hunt and Szymanski [9] reported an $O((n+\ell) \log \ell)$ time bound, where ℓ is the size of the sequence L. This is a significant improvement over the plain dynamic programming algorithm, which always requires $O(n^2)$ time. Although in the worst case ℓ may be $O(n^2)$, in general it may be significantly smaller. The original complexity was not always faster than the plain algorithm, because ℓ may be $\Omega(n/\log n)$. This issue was addressed by Apostolico [1] which obtained $O(n^2)$ time worst case guarantees. Their algorithm already considered using finger trees to represent the T_k lists. Improvements of the Hunt-Szymanski algorithm based on bitwise operations where proposed by Crochemore, Iliopoulous, and Pinzon [5].

A data structure that supports dynamic longest increasing subsequences was presented by Chen, Chu, and Pinsker [4]. The focus is

in supporting insertions anywhere in the sequence, which is achieved in $O(1 + \lambda \log(\ell/\lambda))$ time. The authors obtain one corresponding LIS in $O(\lambda + \log \ell)$ time. This is more efficient than the procedure we explain before Theorem 1, however our procedure can be used to obtain all the subsequences, whereas their approach obtains only one. Their data structure is similar to the one we present, which is expected as both are related to the Hunt-Szymanski algorithm. Chen et al. [4] use level key lists L_k, which are similar to our T_k lists, but store index value pairs and are sorted by increasing index. This is similar to the structure we use for the `GetLIS` operation, but the lists are flattened, instead of storing the indexes in a second structure. Moreover they also use red-black trees to split and concatenate lists and also mention exploring fingering properties of the structure. The presentation mentions deletions but the focus is on insertions. It seems plausible their representation could also support deletions efficiently.

The most recent approach for computing the LTSS was proposed by Inoue, Inenaga, and Bannai [10]. Their algorithm is very similar to the one we present in this paper. They also reduce the problem to a dynamic LIS problem and used the data structure of Chen, Chu, and Pinsker [4] to obtain a complexity of $O(\min\{n, \ell\}\lambda(1 + \log(\ell/\lambda)) + n + \ell \log n)$. In the next section we explain how our algorithm improves upon their result and discuss future possible improvements.

6 Conclusions

In this section we recall and discuss the contributions of the paper in context. In this paper we presented a new algorithm to determine the longest tandem scattered sub-sequence of a string F. In the process we introduced the decremental string comparison problem and provided new data structures to support dynamic LIS sequences. We studied a dynamic version of the Hunt-Szymanski algorithm, which yielded several interesting results.

Considering the LTSS problem itself the strongest work case bounds where obtained by Tiskin [20] with an $O(n^2(\log\log n)^2/(\log n)^2)$ time bound. Both this algorithm and the one by Kosowski [15] seem to have the average case with the same bound as the worst case.

The algorithm we obtain as a consequence of Theorem 2 obtains $O(\min\{n, \ell\}\lambda(1 + \log(\min\{\lambda, \ell/\lambda\})) + n + \ell)$ time, where $\lambda \geq 2$ is the size of the LTSS and $\ell \geq 2$ is the number of pairs of positions in F that contain the same letter. Hence when $\ell = o(n^2(\log\log n)^2/(\log n)^2)$ and

$\lambda = o(n(\log\log n)^2/((\log(\min\{\lambda, \ell/\lambda\}))(\log n)^2))$ our algorithm becomes more efficient. Thus our algorithm is most efficient when the size of the LTSS is small. The extreme case in favor of our algorithm occurs when all the letters in F are distinct. In this case our algorithm is actually linear, i.e., $O(n)$ time and space. This particular case is trivial but a similar situation occurs when the alphabet size is large, i.e., polylog. This is also the case were the original Hunt-Szymanski algorithm obtains its best performance.

One important contribution of this work is the relation between the LTSS and the decremental and incremental string comparison algorithms. This relation is straightforward but seems to have remained unnoticed[4], as the algorithm of Ishida et al. [11] also depends on λ and could thus be used to compute the LTSS, if the structure was also decremental. On the other hand the structure of Kim and Park [14] is decremental but does not depend on λ. This relation was indeed explored in the work of Tiskin [20], but as mentioned above the resulting algorithm is also not dependent on λ. Hence for the case of a single string the algorithm we presented in Section 3 yields competitive results. In fact it is very interesting to compare the T_k lists of the Hunt-Szymanski algorithm to the incremental data structure of Ishida et al. [11]. In essence their structure consists in expanded T_k lists, where each element is repeated several times so that the list becomes size n. This increases the space requirements but makes navigating the lists and across lists more convenient. Also it forces one of the operations be $O(n)$ and thus the overall bound is always $O(n^2)$ instead of $O(n\lambda)$.

The recent work by Inoue, Inenaga, and Bannai [10] follows essentially the same approach as this paper. It solves the LTSS problem by resorting to the same decremental string comparison approach and solves this problem using the Hunt-Szymanski reduction to a LIS problem. A dynamic version of the LIS problem that supports the `ExtractMin` operation is also considered. In fact we were unaware of the similarity of their approach until recently. Still our approach contains several key insights which allow us to obtain a result that is competitive against their $O(\min\{n, \ell\}\lambda(1+\log(\ell/\lambda))+n+\ell\log n)$ time bound[5]. They use essentially the dynamic LIS structure of Chen, Chu, and Pinsker [4] and propose only one improvement, batched `ExtractMin` operations. This means that they obtain good performance for a sequence of `ExtractMin` operations.

[4]It was also recently pointed out by Inoue, Inenaga, and Bannai [10].

[5]Note that we added an $O(n)$ term to their complexity result, because in the case that we considered when all the letters of F are distinct we have $\ell = \lambda = 0$, but their algorithm still requires $O(n)$ time, as does ours.

We obtain the same improvement by using our duplicate discarding approach. Therefore a single `ExtractMin` operation on our data structure corresponds to several on theirs, because their `ExtractMin` operation removes one duplicate at a time whereas ours removes all. Hence our `ExtractMin` operation requires the same time as their batch of `ExtractMin`. One very important optimization of our approach is using the finger trees to represent the T_k lists which leads to the improved performance of the `AppendBatch` operation, Algorithm 5. This implies that there is no $O(\log n)$ factor associated to the ℓ term in our complexity. This term is most likely to dominate the overall complexity in several interesting cases and in our algorithm it is $O(\ell)$ whereas in theirs it is $O(\ell \log n)$. However this is a tradeoff as we obtain an extra $O(n\lambda)$ term, whereas theirs is only $O(n)$. Ignoring the first term, the resulting comparison is between $O(n + \ell \log n)$ for their algorithm and $O(n\lambda + \ell)$ for ours. Hence our algorithm obtains better performance provided that $\ell = \Omega(n(\lambda - 1)/((\log n) - 1))$ and $n \leq \ell$ because of the first term. Let us consider a very simple example where this is likely to happen.

Assume that the letters of F are obtained independently and uniformly at random from an alphabet of size σ. In this case ℓ is expected to be $(n/\sigma)^2$ and the distribution is highly concentrated around this value. Hence in order for our algorithm to obtain the best theoretical bound it is necessary to have an alphabet size σ that is smaller than $\sqrt{(n((\log n) - 1))/(\lambda - 1)}$. This is actually a very loose bound, much larger than poly logarithmic alphabets. Hence our algorithm's theoretical bound yields the best performance for most alphabets, except for exceedingly large ones. Even in extremely large alphabets there is the mitigating expectation that σ and λ have an inverse relation, meaning that larger values of σ should yield smaller values of λ.

The final improvement on the work of Inoue, Inenaga, and Bannai [10] is the analysis with Lagrange multipliers that yields the $\log(\min\{\lambda, \ell/\lambda\})$ bound that improves on the previous $\log(\ell/\lambda)$ complexity. Also we believe that future research on these data structures will focus precisely on this factor. One approach that seems promising is to store the T_k lists in a data structure that supports the dynamic fractional cascading technique of Chazelle and Guibas [3], potentially reducing this factor to $O(\log \log n)$.

The final contribution of this paper is the data structure to maintain a dynamic LIS. Our approach uses a couple of nuances that allow us to obtain Theorem 2. Using the `AppendBatch` of Algorithm 5 the complexity of the original Hunt-Szymanski algorithm drops to $O(\ell + n\lambda)$, which is never more than $O(n^2)$ and sometimes much better. Given the importance of this

algorithm, similar improvements have already been proposed by [1]. Still our work provides a fairly simple alternative.

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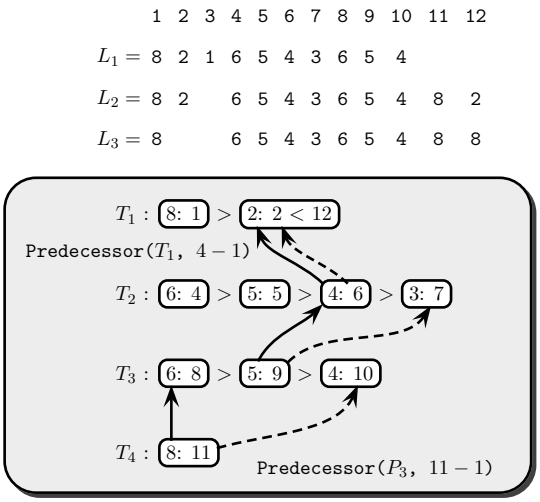
A LIS Retrieval

To simplify the exposition and the analysis we have thus far omitted how to retrieve the actual LIS. This process can be supported by augmenting our data structure, which we will now explain how.

Recall the L sequences that occur in our running example. In the top of Figure 5 we show these sequences, numbered L_1, L_2 and L_3, corresponding to the pairs of strings (P, S), (P', S') and (P'', S''). To retrieve the elements from the list L we need to index them. For L_1 this is straightforward to obtain, we simply number the elements from 1 to 10. However when L_1 changes to L_2 and the number 1 is removed, we do not re-index the sequence. A gap is left at position 3. Likewise when L_2 changes to L_3 a gap is left at position 2. Position 12 can be re-used because it was the last position of L_2.

To retrieve the sequences we augment the elements inside each T_k. Each element stores a value of i of L and a list which contains positions where i occurs. These lists must contain at least one such position, but may contain more than one. The lists contain other positions precisely to avoid repeated elements in a T_k. To support LIS retrieval duplicated elements are not dropped, instead their positions are stored in these lists. Figure 5 shows this structure for L_2 where the list T_1 contains the element 2 and the position list $2, 12$. Note that these position lists can be stored in increasing order,



T_1	T_2	T_3	T_4
2:2,	5:5,	6:8,	8:11
2:2,	4:6,	6:8,	8:11
2:2,	3:7,	6:8,	8:11
2:2,	4:6,	5:9,	8:11
2:2,	3:7,	5:9,	8:11
2:2,	3:7,	4:10,	8:11

Figure 5: Top: Example dynamic list L. Middle: Augmented threshold data structure for L_2. Bottom: Longest increasing sequences for L_2 in the order produced by Algorithm 6.

for each element. Moreover the concatenation of these lists, for a fixed T_k, is also sorted in increasing order. We refer to these global lists as P_k. Hence for L_2 we have $P_1 = 1, 2, 12$ and $P_2 = 4, 5, 6, 7$ and $P_3 = 8, 9, 10$ and $P_4 = 11$.

The possible sequences for our problem are shown in the bottom part of Figure 5. Each sequence is obtained by choosing one element from T_1, T_2, T_3 and T_4, in general one element from T_1 to T_λ. The sequences must be increasing in the values chosen from T_k and also in the values chosen from P_k. To guarantee that searching through these lists always yields a sequence of size λ the procedure starts with $k = \lambda$ and proceeds to decrease k. This is illustrated by the big arrow in Figure 5.

In our example we start at T_4 and choose the value 8 with corresponding

position 11. Now we aim to determine which elements of T_3 may occur in LIS sequences that terminate at 8. To determine the first such element we can compute `Predecessor(`T_3`,` $8-1$`)`, i.e., find the largest value in T_3 that is strictly smaller than 8. Likewise the last such element in P_3 should be `Predecessor(`P_3`,` $11-1$`)`, i.e., it must occur in a position strictly smaller than 11. Recall that T_3 is stored in decreasing order and P_3 in increasing order. Therefore these predecessors define the interval of valid element choices for a LIS. In general the interval of interest for a given element (i, p) of T_{k+1} is between `Predecessor(`T_k`,` $i-1$`)` and `Predecessor(`P_k`,` $p-1$`)`. The `Predecessor` on T_k is obtained from the BST for T_k in $O(\log \ell)$ time. The `Predecessor` on P_k is conceptual and it is enough to verify that $p' < p$, where p' is the current position in P_k. We illustrate these operations with arrows in Figure 5. The dashed lines are used for P_k and the filled lines for T_k. The figure also illustrates the interval for element 5 of T_3, i.e., `Predecessor(`T_2`,`$5-1$`)` and `Predecessor(`P_2`,`$9-1$`)`. Moreover it also shows the interval for element 4 of T_2, i.e., `Predecessor(`T_1`,`$4-1$`)` and `Predecessor(`P_1`,`$6-1$`)`. Therefore, iterating the `Predecessor(`T_k`,` $i-1$`)` operations, we can obtain the lexicographically largest LIS in $O(\lambda \log \ell)$. To obtain the remaining LIS we traverse these intervals, yielding a new LIS for each position that is visited. Algorithm 6 details this procedure.

The arguments for the RecursiveGetLIS are respectively, a stack S, which starts empty, a value for k, a value i of L in T_{k+1} and a corresponding position p in P_{k+1}. The `Predecessor` operation is extended to return the positions p, besides the i values. Since the operation is on T_k it returns the smallest p for the corresponding i, in our example `Predecessor(`T_1`,` $4-1$`)` returns $(2, 2)$ instead of $(2, 12)$. If the corresponding T_k is empty then it returns $(-\infty, +\infty)$. Moreover for this algorithm we also use the `Next` operation, which behaves as an iterator and returns an (i', p') pair. It returns the next element, for example `Next(`P_1`)`, returns $(2, 12)$, assuming it is the first invocation after `Predecessor(`T_1`,` $4-1$`)`. If there is no such element it returns $(+\infty, +\infty)$. Assume that T_k is represented as a BST and P_k is divided into lists, each inside a node of the BST as shown in the middle of Figure 5. The `Next` operation either moves to the next element in the current list, or to the next node on the BST, when it reaches the end of the current list. Note that by next on the BST we mean a smaller value of i, as the T_k are stored in decreasing order. Moving to the next element on a list requires constant time, but finding the next element on the BST may require $O(\log \ell)$ time. Hence Algorithm 6 obtains each LIS in

Algorithm 6 `GetLIS()`

Require: L is not empty

```
1: RecursiveGetLIS(∅,λ + 1, +∞, +∞)

2: procedure RecursiveGetLIS(S, k, i, p)
3:     if k = 0 then
4:         return S                                   ▷ Found a LIS
5:     else
6:         (i′, p′) ← Predecessor(T_k, i − 1)
7:         while p′ < p do
8:             Push(S, p′)
9:             RecursiveGetLIS(S,k − 1,i′,p′)
10:            Pop(S)
11:            (i′, p′) ← Next(P_k)
12:        end while
13:    end if
14: end procedure
```

$O(\lambda \log \ell)$ time, which again can be reduced to $O(1+\lambda \log(\ell/\lambda))$ by Jensen's inequality.

B Correctness

We focused mainly on the complexity of operations `Append` and `ExtractMin`. However for completeness we will now establish that the procedures of Algorithm 3 and Algorithm 4 are correct. Proving the correctness of the `Append` operation per si is not essential as this was the procedure proposed by Hunt and Szymanski [9] for determining a "static" LIS. Hence any sequence of `Append` operations will compute a LIS. All we need to do is to verify the interaction with the `ExtractMin` operation. This leads us to identify the invariant properties of the T_k lists. The first more evident properties are related to the sorted order of the T_k lists.

Lemma 1 *Each T_k list stores its elements sorted in decreasing order.*

These Lemmas are proven using an argument of structural induction, meaning that we assume that the property holds before a given application of a `Append` or `ExtractMin` operation and only need to prove that after

the operation terminates the property is valid in the resulting data structure. We use superscript b's to represent the structure before the operation, meaning that a T_k^b list represents a list before the operation is applied. The list after the operation is applied is simply a T_k list. Also, for an element i in T_k we say that it is at level k.

Proof: Let us separate the analysis into the two operations:

`Append`**:** before the element i is added to list T_k in line 12 a binary search is used to determine k. This process guarantees that i is smaller than or equal to $\texttt{Min}[k]$, i.e., smaller than or equal to the last element of T_k. Our discarding approach means that if is equal it gets discarded. Therefore the list are kept in strictly decreasing order. The order of the other lists remains unaltered.

`ExtractMin`**:** a T_{k-1} list that results from this operation consists in the concatenation of a prefix of the previous T_{k-1}^b list with a suffix of a T_k^b list. This process is executed in lines 7 and 8 of Algorithm 4. The main observation here is that the prefix of T_{k-1}^b ends in a value n_{k-1}, computed by $\texttt{Min}(T_{k-1})$ and the suffix starts at a value v_k, computed in line 6 and the `Predecessor` operation in this line guarantees that $n_{k-1} \geq v_k$. Note that in our algorithm if $n_{k-1} = v_k$ the value v_k gets discarded. This means the overall list preserves its strictly decreasing order.

□

Lemma 2 *The sequence of the minimum elements of the T_k lists is sorted in increasing order.*

Proof: Again let us separate the analysis into the two operations:

`Append`**:** before the element i is added to list T_k in line 12 a binary search is used to determine k. This process guarantees that i is strictly larger than $\texttt{Min}[k-1]$, i.e., larger than the last element of T_{k-1}. Moreover it also guarantees that i is smaller than or equal to $\texttt{Min}[k]$, which by hypothesis is strictly smaller than $\texttt{Min}[k+1]$. Therefore the increasing order of the last elements is maintained. The order of remaining last elements is preserved.

`ExtractMin`**:** this operation does not alter the relative order of the last elements. It divides the T_k^b lists and moves a suffix of T_k^b to T_{k-1},

this means that the minimum elements simply move to a lower level. Hence we are only interested in the level k where this process finishes, i.e., $T_k^b = T_k$. Which means that the minimum element of this level gets preserved. Moreover for all $k' \geq k$ their minimum elements also get preserved and therefore their relative order remains an increasing sequence. Hence we only need to verify two minimum relations between T_k and T_{k-1} and between T_{k-1} and T_{k-2}. Note that T_{k-1} is only a prefix of T_{k-1}^b, because it does not pull elements from T_k^b this means that the minimum of T_{k-1} and the minimum of T_{k-2} are actually two elements that occurred in the same T_{k-1}^b. Because of Lemma 1 this implies that the minimum of T_{k-2} is strictly smaller than the minimum of T_{k-1}, because the latter occurs first in T_{k-1}^b. Hence we are left with verifying the relation between the minimum of T_{k-1}^b and the minimum of T_k^b. This is precisely the relation that is used in the **while** guard in line 5. Because we are considering the position where the process terminates we have that the `while` guard is false which implies that $\texttt{Min}(T_{k-1}) < \texttt{Min}(T_k)$. Hence the overall sequence of minimum elements of the T_k lists remains sorted in increasing order.

□

From Lemma 1 we can conclude that when an element i is inserted into the list T_k it most be the smallest element on that list. Moreover from both these Lemmas we can conclude that for any index k the minimum element of T_k is strictly smaller than any element in $T_{k'}$ for any level $k' \geq k$. In particular for $k = 1$ this implies that the last element of T_1 is actually the overall minimum.

This is the last of the order properties that is required for our study. Now we need to analyze other properties of the structure. For this particular analysis we strip away the layers of complexity that are related to the fact that the T_k lists are sorted. Instead we use sets of positions. Moreover the T_k lists are actually a compressed version of the position sets P_k, by compressed we mean that our duplicate discarding approach that there might not be a one to one correspondence between T_k and P_k. We will now use the position sets a they provide for simpler notation.

First we define a set P of positions as a finite set of integers. The integers do not have to be consecutive and gaps may occur. For example for the list L_2 in Figure 5 the set P consists of the integers from 1 to 12 except for 3. This set is then partitioned into the P_k sets. Hence the resulting sets

are the following:

$$P_1 = \{1, 2, 12\}$$
$$P_2 = \{4, 5, 6, 7\}$$
$$P_3 = \{8, 9, 10\}$$
$$P_4 = \{11\}$$

In general the P_k sets form a disjoint partition of P meaning that $P_k \cap P_{k'} = \emptyset$ for any indices $k \neq k'$ and $P = P_1 \cup \ldots \cup P_\lambda$. A T_k set can be obtained from the corresponding P_k set by accessing the underlying sequence L, i.e., $T_k = \{L[p] | p \in P_k\}$. We use the notation $L[p]$ to refer to the value in L that corresponds to position p, for example using sequence L_2 we have that $L[12] = 2$. The resulting P_k sets for this example are the following:

$$T_1 = \{8, 2\}$$
$$T_2 = \{6, 5, 4, 3\}$$
$$T_3 = \{6, 5, 4\}$$
$$T_4 = \{8\}$$

We identify two fundamental invariants for the P_k sets. The first concerns the order in the P_k sets and the second is used to guarantee a LIS of size k. These two properties are summarized in the Lemma 3 and Lemma 4.

Lemma 3 *Let $p \in P_k$ and $p' \in P_{k'}$ with $k \leq k'$ be positions from a list L, if $L[p] > L[p']$ then $p < p'$.*

Proof:

`Append`**:** check Algorithm 3. Assume the element being inserted is i. We only need to check the cases when $i = L[p]$ or $i = L[p']$, because if p and p' are any other positions they do not move from P_k and $P_{k'}$ and their previous relation is preserved. Assume first that $i = L[p']$, because the operation is an `Append` we have that p' is the maximum value inside P, therefore $p < p'$ and the implication holds because the consequent is true. Now let us consider the case where $i = L[p]$, this means that p got inserted into P_k and therefore $L[p]$ is the minimum of T_k (Lemma 1). Therefore we have that $L[p] < L[p']$, as was observed just after the proof of Lemma 2. Therefore the antecedent is false and, again, the implication holds true.

`ExtractMin`: check Algorithm 4. There are three relevant cases. Either p was in level $k+1$ before the operation or p' was in level $k'+1$ before or both. Note that the case where both p and p' remain in the same k, i.e., $p \in P_k^b$ and $p' \in P_{k'}'^b$ is trivial.

First consider the cases where p' moves from $k'+1$ to k', if $k < k'+1$ the property is preserved and therefore the implication holds. The only tricky case is when $k = k'+1$. If, on the other hand, p stays fixed at k and p' moves to $k' < k$ then the implication might change but the Lemma still holds because the condition $k \leq k'$ is not satisfied.

Second let us consider the case where p moves from $k+1$ to k, particularly when $k = k'$ and p' stays fixed at k'. This time we can not copy the validity of the implication because it is not true that $k+1 \leq k'$ and therefore the validity of the implication might be false because the Lemma condition was not satisfied. In this case we have that $v_{k+1} \geq L[p]$, where v_k is the value assigned to the variable v in the line 6 of Algorithm 4 when processing list T_k. Moreover, because the `Predecessor` operation is used, we have that $n_k \geq v_{k+1}$ where n_{k-1} is the value `Min`(T_{k-1}) computed in line 6 of Algorithm 4. Moreover because n_k is a minimum and p' remains at level k we have that $L[p'] \geq n_k$. Combining these inequalities yields $L[p'] \geq n_k \geq v_{k+1} \geq L[p]$, which by transitivity implies $L[p'] \geq L[p]$, thus making the antecedent of the implication false and verifying its validity. □

Lemma 4 *For any position $p \in P_k$ with $1 < k$ there exits a position $p' \in P_{k-1}$ such that $p' < p$ and $L[p] > L[p']$.*

Proof:

`Append`: Assume p gets inserted into some P_k with $k > 1$. Moreover let $p' \in P_{k-1}$ be the position for which $L[p']$ is the minimum in T_{k-1}. Then the observation after the proof of Lemma 2 guarantees that $L[p] > L[p']$. Also note that the `Append` operation always inserts the maximal p value and therefore we have trivially that $p' < p$, thus satisfying the conditions in the Lemma.

`ExtractMin`: We have two possibilities either p moved from level $k+1$ to level k or it stayed at level k meaning that it started in set P_k^b and ended in the set P_k.

First consider that p moves from level $k+1$ level k. By hypothesis there a $p' \in P_k^b$ that satisfies $p' < p$ and $L[p] > L[p']$. If this position moves to level $k-1$, i.e., $p' \in P_{k-1}$ then the Lemma is verified. The alternative does not occur, as can be seen by contradiction. Suppose that we have both p and p' in P_k and $L[p] > L[p']$, according to Lemma 3 this implies that $p < p'$ which contradicts the hypothesis that $p' < p$.

Second consider that p stayed at level k, meaning that $L[p] > n_{k-1}$, where n_{k-1} is, as in the previous proofs, the value of $\texttt{Min}(T_{k-1})$ obtained in line 6 of Algorithm 4. This holds precisely because n_{k-1} is used to determine v_k which is used to split T_k^b. Let $p' \in P_{k-1}$ be a position that justifies n_{k-1}, i.e., $L[p'] = n_{k-1}$. Hence we can write that $L[p] > L[p']$. If $p' < p$ then we obtained our desired element. Otherwise we still have by hypothesis that there is a $p'' \in P_{k-1}^b$ such that $p'' < p$ and $L[p] > L[p'']$. The only problem would be if p'' also moved down to level $k-2$. This does not occur, as can be verified by contradiction. If p'' were to move we would have $L[p'] = n_{k-1} > L[p'']$ and would conclude that $p' < p''$ by Lemma 3. We thus apply transitivity to our contradiction hypothesis that $p \leq p'$ and conclude that $p < p''$. This in turn contradicts the hypothesis that $p'' < p$. □

We can now prove that the data structure that we used for Theorem 1 is correct. In essence we only need to show that if list T_k is not empty there the sequence L contains a LIS of size at least k. This is simply a matter of iterating Lemma 4 starting with some position p where $L[p] \in T_k$. This yields a list of k positions $p_k = p, p_{k-1} = p', \ldots, p_1$, such that $p_1 < \ldots < p_{k-1} < p_k$ and $L[p_1] < \ldots < L[p_{k-1}] < L[p_k]$, where both conditions are consequences of the Lemma.