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A Note on Congruences of Infinite Bounded Involution Lattices

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Dedicated to the memory of my beloved grandmother, Elena Mircea

Abstract

We prove that an infinite (bounded) involution lattice and even pseudo–Kleene algebra can have any number of congruences between 2 and its number of elements or equalling its number of subsets, regardless of whether it has as many ideals as elements or as many ideals as subsets. Furthermore, when they have at most as many congruences as elements, these involution lattices and even pseudo–Kleene algebras can be chosen such that all their lattice congruences preserve their involutions, so that they have as many congruences as their lattice reducts. Under the Generalized Continuum Hypothesis, this means that an infinite (bounded) involution lattice and even pseudo–Kleene algebra can have any number of congruences between 2 and its number of subsets, regardless of its number of ideals. Consequently, the same holds for antiortholattices, a class of paraorthomodular Brouwer–Zadeh lattices. Regarding the shapes of the congruence lattices of the lattice–ordered algebras in question, it turns out that, as long as the number of congruences is not strictly larger than the number of elements, they can be isomorphic to any nonsingleton well–ordered set with a largest element of any of those cardinalities, provided its largest element is

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strictly join–irreducible in the case of bounded lattice–ordered algebras and, in the case of antiortholattices with at least 3 distinct elements, provided that the predecessor of the largest element of that well–ordered set is strictly join–irreducible, as well; of course, various constructions can be applied to these algebras to obtain congruence lattices with different structures without changing the cardinalities in question. We point out sufficient conditions for analogous results to hold in an arbitrary variety.

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1 Introduction

As part of our main result from [4], we have proven that, under the Generalized Continuum Hypothesis, an infinite lattice can have any number of congruences between 2 and its number of subsets. In this paper, we prove that the same holds for infinite (bounded) involution lattices and even infinite pseudo–Kleene algebras. Thus the same holds for infinite antiortholattices, which are algebraic structures with pseudo–Kleene algebra reducts originating in the study of quantum logics [6, 7, 8, 9, 10, 15, 16]. Moreover, we can let all these lattice–ordered algebras have any numbers of ideals, while keeping this property on congruences. Furthermore, if we restrict to numbers of congruences that are either smaller than the numbers of elements or equal to the numbers of subsets of these algebras, then we do not need to enforce the Continuum Hypothesis.

To obtain our main theorem: Theorem 2, we extract from our method in [4] the general results that hold in any variety; see Lemma 2 and the paragraph right after this lemma, in which we point out that these properties could be applied in other varieties to obtain similar results; all we need in such a variety $\mathbb{V}$ is some construction as the one from [14] for obtaining, from an infinite algebra A, an algebra B of the same cardinality as A, having one congruence more than A; if we can obtain B such that it includes A and has the congruences of a certain form (see conditions Ⓢ$_{\mathbb{V}}$ and ⓒ$_{\mathbb{V}}$ in Section 5), then the general method from Lemma 2 can be applied directly, so we may conclude that there exist, in $\mathbb{V}$, algebras with the same cardinality as A and with any number of congruences smaller than the cardinality of A and larger than the smallest number of congruences that A can have such that an algebra B as above can exist for that algebra A. The construction we provide produces algebras with certain shapes for their congruence lattices, so a condition as above ensures that such a variety $\mathbb{V}$ has members with those particular congruence lattices (see Proposition 1).

Note that, in the finite case, these results on numbers of congruences do not hold, due to the limited number of configurations. The finite case for lattices has been treated in [2, 17], the one for semilattices in [3], and the one for involution lattices, pseudo–Kleene algebras and antiortholattices in [15].

2 Some Notations

All algebras will be designated by their underlying sets. By *trivial algebra* we mean one–element algebra, and by *simple algebra* we mean algebra with at most two congruences.

We denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. $\amalg$ denotes the disjoint union. For any set M, we denote by $|M|$ the cardinality of M and by $\mathcal{P}(M)$ the set of the subsets of M. If M is a nonempty set, then $\mathrm{Part}(M)$ and $(\mathrm{Eq}(M), \cap, \vee, \Delta_M, \nabla_M)$ will denote the bounded lattices of the partitions and the equivalences of M, respectively, where we keep the order of the lattice operations from [11]; also, we denote by $eq : \mathrm{Part}(M) \to \mathrm{Eq}(M)$ the canonical lattice isomorphism; for any finite partition $\{M_1, \ldots, M_n\}$, $eq(\{M_1, \ldots, M_n\})$ will be streamlined to $eq(M_1, \ldots, M_n)$. If M is an ordered set, then we denote the set of its maximal elements by $\mathrm{Max}(M)$.

Let $\mathbb{V}$ be a variety of algebras of a similarity type τ and A and B be algebras with reducts belonging to $\mathbb{V}$. Following [8, 15], we denote by $A \cong_{\mathbb{V}} B$ the fact that the τ–reducts of A and B are isomorphic, and by $\mathrm{Con}_{\mathbb{V}}(A)$ and $\mathcal{S}_{\mathbb{V}}(A)$ the sets of the congruences and the subalgebras of the τ–reduct of A, respectively. Obviously, for any $\theta \in \mathrm{Con}_{\mathbb{V}}(A)$ and any $S \in \mathcal{S}_{\mathbb{V}}(A)$, we have $\theta \cap S^2 \in \mathrm{Con}_{\mathbb{V}}(S)$. If $\mathbb{V}$ is the variety of lattices or that of bounded lattices, then we eliminate the index $_{\mathbb{V}}$ from the previous notations.

Now let σ be a similarity type of reducts of τ–algebras and $\mathbb{W}$ be a variety of algebras of type σ. Recall from [12, Corollary 2, p. 51] that $\mathrm{Con}_{\mathbb{V}}(A)$ is a complete sublattice of $\mathrm{Eq}(A)$, from which it immediately follows that $\mathrm{Con}_{\mathbb{V}}(A)$ is a complete bounded sublattice of $\mathrm{Con}_{\mathbb{W}}(A)$. In particular, if $\mathbb{V}$ is a variety of algebras with lattice reducts, then $\mathrm{Con}_{\mathbb{V}}(A)$ is a complete bounded sublattice of $\mathrm{Con}(A)$.

If $n \in \mathbb{N}^*$ and the type τ contains constants $\kappa_1, \ldots, \kappa_n$, then we denote by $\mathrm{Con}_{\mathbb{V}_{\kappa_1 \ldots \kappa_n}}(A) = \{\theta \in \mathrm{Con}_{\mathbb{V}}(A) : \kappa_1^A/\theta = \{\kappa_1^A\}, \ldots, \kappa_n^A/\theta = \{\kappa_n^A\}\}$: the set of the congruences of A with the classes of $\kappa_1^A, \ldots, \kappa_n^A$ singletons. It is easy to see that $\mathrm{Con}_{\mathbb{V}_{\kappa_1 \ldots \kappa_n}}(A)$ is a complete sublattice of $\mathrm{Con}_{\mathbb{V}}(A)$ and thus a bounded lattice [9, 15].

Let L be a (bounded) lattice. Then the dual of L will be denoted by L^d.

For any $X \subseteq L$ and any $a, b \in L$, we denote by $[X)_L$ and $[a)_L$ the filter of L generated by X and by a, respectively, and by $(X]_L$ and $(a]_L$ the ideal of L generated by X and by a, respectively; we also denote $[a, b]_L = [a)_L \cap (b]_L$ and $[a, b)_L = [a, b]_L \setminus \{b\}$.

The sets of the filters, principal filters, ideals and principal ideals of L will be denoted by $\mathrm{Filt}(L)$, $\mathrm{PFilt}(L)$, $\mathrm{Id}(L)$ and $\mathrm{PId}(L)$, respectively. Of course, we have: $\mathrm{Con}(L^d) = \mathrm{Con}(L)$, $\mathrm{Filt}(L^d) = \mathrm{Id}(L)$ and $\mathrm{Id}(L^d) = \mathrm{Filt}(L)$.

Recall that the prime ideals of L are exactly the set complements of its prime filters. Moreover, if C is a chain, then all its proper filters are prime and the same goes for its ideals, hence the proper ideals of C are exactly the set complements of its proper filters, in particular $|\mathrm{Filt}(C)| = |\mathrm{Id}(C)|$. For any $n \in \mathbb{N}^*$, $\mathcal{C}_n$ will denote the n–element chain.

If the lattice L has a 0, then L is said to be 0*–regular* iff, for all $\theta, \zeta \in \mathrm{Con}(L)$, $0/\theta = 0/\zeta$ implies $\theta = \zeta$.

Recall that L satisfies the *Ascending*, respectively the *Descending Chain Condition* (abbreviated *ACC* and *DCC*, respectively) iff, for any sequence $(x_n)_{n \in \mathbb{N}}$ of elements of L such that $x_k \leq x_{k+1}$, respectively $x_k \geq x_{k+1}$ for all $k \in \mathbb{N}$, there exists an $m \in \mathbb{N}$ such that $x_k = x_{k+1}$ for all $k \geq m$. Clearly, any lattice that satisfies the ACC has all ideals principal and, dually, any lattice satisfying the DCC has all filters principal.

3 The Algebras We Are Working With and the Basic Constructions of Ordinal and Horizontal Sum

Definition 1 *We call a* lattice with involution *or* involution lattice *(in brief,* i–lattice*) an algebra* $(L, \wedge, \vee, \cdot')$ *of type* $(2, 2, 1)$*, where* $(L, \wedge, \vee)$ *is a lattice and* $\cdot'$ *is an order–reversing operation such that* $a'' = a$ *for all* $a \in L$*, called* involution*.*

A bounded involution lattice *(in brief,* bi–lattice*) is an algebra* $(L, \wedge, \vee, \cdot', 0, 1)$ *of type* $(2, 2, 1, 0, 0)$*, where* $(L, \wedge, \vee, 0, 1)$ *is a bounded lattice and* $(L, \wedge, \vee, \cdot')$ *is an i–lattice.*

Distributive bi–lattices are called De Morgan algebras*.*
We consider the following condition on a bi–lattice L:

$$\text{ⓚ} \quad \textit{for all } a, b \in L,\ a \wedge a' \leq b \vee b'$$

A pseudo–Kleene algebra *is a bi–lattice that satisfies condition* ⓚ*. The involution of a pseudo–Kleene algebra is called* Kleene complement*.*

Distributive pseudo–Kleene algebras (that is De Morgan algebras which satisfy condition ⓚ*) are called* Kleene algebras *or* Kleene lattices*.*

Linearly ordered (bounded) involution lattices are called (bounded) involution chains*, abbreviated as in the case of arbitrary lattices above; linearly ordered Kleene lattices are called* Kleene chains*; the same goes for the lattice–ordered structures in the next definition.*

A bi–lattice L is said to be paraorthomodular *iff, for all $a, b \in L$, if $a \leq b$ and $a' \wedge b = 0$, then $a = b$.*

A bi–lattice L is said to be orthomodular *iff, for all $a, b \in L$, if $a \leq b$, then $a \vee (a' \wedge b) = b$.*

Of course, any orthomodular lattice L is a paraorthomodular pseudo–Kleene algebra (having $a \wedge a' = 0$ and $a \vee a' = 1$ for all $a \in L$).

We will denote by $\mathbb{I}$, $\mathbb{BI}$ and $\mathbb{KL}$ the variety of involution lattices, bounded involution lattices and pseudo–Kleene algebras, respectively.

An i–lattice with underlying set L and involution $\cdot'$ will often be designated by $(L, \cdot')$. Unless specified otherwise, the involution of an i–lattice will be denoted $\cdot'$. Obviously, the involution of any i–lattice L is a dual lattice automorphism of L, hence L is self–dual and thus it has $|\text{Filt}(L)| = |\text{Id}(L)|$. Regarding congruences:

Remark 1 *Clearly, for any bi–lattice L, $\text{Con}_{\mathbb{BI}01}(L) = \text{Con}_{\mathbb{BI}0}(L)$. Moreover, for any variety $\mathbb{V}$ whose members have bi–lattice reducts and any member A of $\mathbb{V}$, we have $\text{Con}_{\mathbb{V}01}(A) = \text{Con}_{\mathbb{V}0}(A)$.*

We will often use the remarks in this paper without referencing them.

Of course, any Boolean algebra A is a Kleene lattice with the involution equalling its Boolean complement, and, since the Boolean complement is preserved by all lattice congruences, we have $\text{Con}_{\mathbb{I}}(A) = \text{Con}(A)$. Remember that Boolean algebras are exactly the distributive orthomodular lattices. Furthermore, any orthomodular lattice L is a paraorthomodular pseudo–Kleene algebra with all its lattice congruences preserving its involution, so that $\text{Con}_{\mathbb{I}}(L) = \text{Con}(L)$ [1].

Definition 2 [6, 7, 8, 9, 10, 16] *A* Brouwer–Zadeh lattice *(in brief,* BZ–lattice*) is an algebra* $(L, \wedge, \vee, \cdot', \cdot^{\sim}, 0, 1)$ *of type* $(2, 2, 1, 1, 0, 0)$ *such that* $(L, \wedge, \vee, \cdot', 0, 1)$ *is a pseudo–Kleene algebra and the unary operation* $\cdot^{\sim}$*, called* Brouwer complement*, is order–reversing and satisfies* $a \wedge a^{\sim} = 0$ *and* $a \leq a^{\sim\sim} = a^{\sim\prime}$ *for all* $a \in L$*.*

The Brouwer complement on a BZ–lattice L *defined by* $0^{\sim} = 1$ *and* $a^{\sim} = 0$ *for all* $a \in L \setminus \{0\}$ *is called the* trivial Brouwer complement*.*

The following equation in the language of BZ–lattices is called the Strong De Morgan condition (SDM): $(x \wedge y)^{\sim} \approx x^{\sim} \vee y^{\sim}$*.*

A PBZ*–lattice *is a paraorthomodular BZ–lattice* L *that satisfies the following weakening of the SDM:* $(x \wedge x')^{\sim} \approx x^{\sim} \vee x'^{\sim}$*.*

An antiortholattice *is a PBZ***–lattice with the property that* 0 *and* 1 *are its only elements whose Kleene complements are bounded lattice complements.*

We denote by $\mathbb{BZL}$ the variety of BZ–lattices. PBZ*–lattices form a variety, as well, but antiortholattices form a proper universal class [6, 7, 8, 9, 10, 16].

Antiortholattices are exactly the PBZ*–lattices whose Brouwer complement is trivial. An antiortholattice satisfies the SDM iff it has the 0 meet–irreducible. Moreover, any pseudo–Kleene algebra with the 0 meet–irreducible, endowed with the trivial Brouwer complement, becomes an antiortholattice (which, of course, satisfies the SDM). See a strenghthening of the latter property in the next section, and [6, 7, 8, 9, 10, 16] for all these properties.

We now recall the definition of the horizontal sum of a family of nontrivial bounded lattices, obtained by glueing those lattices at their bottom elements and at their top elements. Let $(L_i, \leq^{L_i}, 0^{L_i}, 1^{L_i})_{i \in I}$ be a nonempty family of nontrivial bounded lattices. Then the *horizontal sum* of the family $(L_i, \leq^{L_i}, 0^{L_i}, 1^{L_i})_{i \in I}$ is the bounded lattice $(\boxplus_{i \in I} L_i, \leq, 0, 1)$ defined in this way: let $L = \amalg_{i \in I} L_i$ and ε the equivalence on L that collapses only the bottom elements of these lattices, as well as their top elements: $\varepsilon = eq(\{\{0^{L_i} : i \in I\}, \{1^{L_i} : i \in I\}\} \cup \{\{x\} : x \in L \setminus \{0^{L_i}, 1^{L_i} : i \in I\}\}) \in \mathrm{Eq}(L)$; denote by $0 = 0^{L_i}/\varepsilon$ and $1 = 1^{L_i}/\varepsilon$ for some $i \in I$; then, for every $i \in I$, $\varepsilon \cap L_i^2 = \Delta_{L_i} \in \mathrm{Con}(L_i)$, so $L_i \cong L_i/\varepsilon$; we identify each L_i with L_i/ε, by identifying x with x/ε for all $x \in L$, thus obtaining $0 = 0^{L_i}$ and $1 = 1^{L_i}$ for all $i \in I$; now we set $\boxplus_{i \in I} L_i = L/\varepsilon$ and $\leq = \bigcup_{i \in I} \leq^{L_i}$.

Clearly, the horizontal sum as a binary operation between nontrivial bounded lattices is associative and commutative. We may also note that, for any nontrivial bounded lattice L, we have $\mathcal{C}_2 \boxplus L = L$, so we obtain a nontrivial horizontal sum (that is a horizontal sum which strictly includes each of its summands) iff we have at least two summands of cardinality strictly greater than 2.

We denote by $\mathcal{M}_{|I|} = \boxplus_{i\in I}\mathcal{C}_3$ the modular lattice of length 3 and cardinality $|I|+2$, which is clearly simple.

If $(L_i, \cdot^{\prime i})_{i\in I}$ is a nonempty family of nontrivial bi–lattices, then the *horizontal sum* of this family is the bi–lattice $(\boxplus_{i\in I} L_i, \cdot^{\prime})$, whose underlying bounded lattice is the horizontal sum of the family of the bounded lattice reducts $(L_i)_{i\in I}$ and whose involution is defined by: $\cdot^{\prime} \mid_{L_i} = \cdot^{\prime i}$ for all $i \in I$.

See in [14, 15] the congruence lattice of any horizontal sum of nontrivial bounded lattices or bi–lattices.

Let $(L, \leq^L)$ be a lattice with top element 1^L and $(M, \leq^M)$ a lattice with bottom element 0^M. Recall that the *ordinal sum* of L with M is the lattice $(L \oplus M, \leq)$ obtained by glueing the top element of L and the bottom element of M together, thus stacking M on top of L. For the precise definition, let ε be the equivalence on $L \amalg M$ that only collapses 1^L with 0^M: $\varepsilon = eq(\{\{1^L, 0^M\}\} \cup \{\{x\} : x \in L \amalg M \setminus \{1^L, 0^M\}\}) \in \mathrm{Eq}(L \amalg M)$. We note that $\varepsilon \cap L^2 = \Delta_L \in \mathrm{Con}(L)$ and $\varepsilon \cap M^2 = \Delta_M \in \mathrm{Con}(M)$, so that we may identify L with L/ε and M with M/ε by identifying each $x \in L \amalg M$ with x/ε. Now we let $L \oplus M = (L \amalg M)/\varepsilon$ and $\leq = \leq^L \cup \leq^M \cup \{(x, y) : x \in L, y \in M\}$.

Note that $\mathrm{Filt}(L \oplus M) = \mathrm{Filt}(M) \cup \{F \cup L : F \in \mathrm{Filt}(L)\}$ and $\mathrm{Id}(L \oplus M) = \mathrm{Id}(L) \cup \{L \cup I : I \in \mathrm{Id}(M)\}$, thus $|\mathrm{Filt}(L \oplus M)| = |\mathrm{Filt}(L)| + |\mathrm{Filt}(M)| - 1$ and $|\mathrm{Id}(L \oplus M)| = |\mathrm{Id}(L)| + |\mathrm{Id}(M)| - 1$, where we let $\kappa - \lambda = \kappa$ for any infinite cardinal number κ and any cardinal number $\lambda < \kappa$.

If, for every $\alpha \in \mathrm{Con}(L)$ and every $\beta \in \mathrm{Con}(M)$, we denote by $\alpha \oplus \beta$ the equivalence on $L \oplus M$ whose classes are those of α and β, excepting the classes of the common element $1^L = 0^M$ of L and M in $L \oplus M$ modulo α, respectively β, along with the union of latter two classes: $\alpha \oplus \beta = eq((L/\alpha \setminus 1^L/\alpha) \cup (M/\beta \setminus 0^M/\beta) \cup \{1^L/\alpha \cup 0^M/\beta\})$, then, clearly, $\alpha \oplus \beta \in \mathrm{Con}(L \oplus M)$. Furthermore, since L and M are sublattices of $L \oplus M$, for every $\theta \in \mathrm{Con}(L \oplus M)$, we have $\theta \cap L^2 \in \mathrm{Con}(L)$ and $\theta \cap M^2 \in \mathrm{Con}(M)$, and clearly $\theta = (\theta \cap L^2) \oplus (\theta \cap M^2)$. Therefore the map $(\alpha, \beta) \mapsto \alpha \oplus \beta$ is a lattice isomorphism from $\mathrm{Con}(L) \times \mathrm{Con}(M)$ to $\mathrm{Con}(L \oplus M)$.

Clearly, the ordinal sum of bounded lattices is associative and so is the operation $\oplus$ on congruences of those bounded lattices.

4 Some Particular Constructions of Lattices and Lattice–ordered Algebras and Their Congruences

Let L be a lattice with top element, $f : L \rightarrow L^d$ a dual lattice isomorphism and $(K, \cdot'^K)$ a bi–lattice. Then $L \oplus K \oplus L^d$, and in particular $L \oplus L^d$ in the case when K is the one–element chain, becomes an i–lattice with the involution $\cdot' : L \oplus K \oplus L^d \rightarrow L \oplus K \oplus L^d$ defined by: $\cdot' \mid_L = f$, $\cdot' \mid_K = \cdot'^K$ and $\cdot' \mid_{L^d} = f^{-1}$. Whenever we refer to a bi–lattice with a bounded lattice reduct of this form: $L \oplus K \oplus L^d$, we consider it endowed with the involution from this canonical construction.

Notice that, if L is a bounded lattice and K is a pseudo–Kleene algebra, then $L \oplus K \oplus L^d$ satisfies ⓚ, thus $L \oplus K \oplus L^d$ is a pseudo–Kleene algebra. Moreover, it is straightforward that, if L is a nontrivial bounded lattice and $K \in \mathbb{KL}$, then the pseudo–Kleene algebra $L \oplus K \oplus L^d$ becomes an antiortholattice when endowed with the trivial Brouwer complement [9, 16, 10]. In particular, for any bounded lattice L, $L \oplus L^d$ is a pseudo–Kleene algebra, which becomes an antiortholattice when endowed with the trivial Brouwer complement.

Remark 2 [14, 9, 16, 10, 17, 15] *For any i–lattice $(A, \cdot')$, if we denote by $U' = \{(a', b') : (a, b) \in U\}$ for all $U \subseteq A^2$, then we clearly have $\mathrm{Con}_{\mathbb{I}}(A) = \{\theta \in \mathrm{Con}(A) : \theta = \theta'\}$.*

From this it is immediate that, for any bounded lattice L and any bi–lattice K, $\mathrm{Con}_{\mathbb{I}}(L \oplus K \oplus L^d) = \{\alpha \oplus \beta \oplus \alpha' : \alpha \in \mathrm{Con}(L), \beta \in \mathrm{Con}_{\mathbb{I}}(K)\} \cong \mathrm{Con}(L) \times \mathrm{Con}_{\mathbb{I}}(K)$, in particular $\mathrm{Con}_{\mathbb{I}}(L \oplus L^d) = \{\alpha \oplus \alpha' : \alpha \in \mathrm{Con}(L)\} \cong \mathrm{Con}(L)$.

Remark 3 [7, 9, 15] *It is routine to prove that, for any antiortholattice A, $\mathrm{Con}_{\mathbb{BZL}}(A) = \mathrm{Con}_{\mathbb{BZL}0}(A) \cup \{\nabla_A\} = \mathrm{Con}_{\mathbb{BI}0}(A) \cup \{\nabla_A\} \cong \mathrm{Con}_{\mathbb{BI}0}(A) \oplus \mathcal{C}_2$.*

Therefore, if L is a nontrivial bounded lattice and $K \in \mathbb{KL}$, then the antiortholattice $L \oplus K \oplus L^d$ has $\mathrm{Con}_{\mathbb{BZL}}(L \oplus K \oplus L^d) = \mathrm{Con}_{\mathbb{BI}0}(L \oplus K \oplus L^d) \cup \{\nabla_{L \oplus K \oplus L^d}\} = \{\alpha \oplus \beta \oplus \alpha' : \alpha \in \mathrm{Con}_0(L), \beta \in \mathrm{Con}_{\mathbb{I}}(K)\} \cup \{\nabla_{L \oplus K \oplus L^d}\} \cong (\mathrm{Con}_0(L) \times \mathrm{Con}_{\mathbb{I}}(K)) \oplus \mathcal{C}_2$.

Thus, if L is 0–regular, so that $\mathrm{Con}_0(L) = \{\Delta_L\}$, then $\mathrm{Con}_{\mathbb{BZL}}(L \oplus K \oplus L^d) = \{\Delta_L \oplus \beta \oplus \Delta_{L^d} : \beta \in \mathrm{Con}_{\mathbb{I}}(K)\} \cup \{\nabla_{L \oplus K \oplus L^d}\} \cong \mathrm{Con}_{\mathbb{I}}(K) \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{BZL}}(L \oplus K \oplus L^d)| = |\mathrm{Con}_{\mathbb{I}}(K)| + 1$.

In particular, $\mathrm{Con}_{\mathbb{BZL}}(\mathcal{C}_2 \oplus K \oplus \mathcal{C}_2) = \{eq(K/\beta \cup \{\{0\}, \{1\}\}) : \beta \in \mathrm{Con}_{\mathbb{I}}(K)\} \cup \{\nabla_{\mathcal{C}_2 \oplus K \oplus \mathcal{C}_2}\} \cong \mathrm{Con}_{\mathbb{I}}(K) \oplus \mathcal{C}_2$, thus $|\mathrm{Con}_{\mathbb{BZL}}(\mathcal{C}_2 \oplus K \oplus \mathcal{C}_2)| =$

$|\mathrm{Con}_{\mathbb{I}}(K)|+1$ *and, for any* 0*–regular nontrivial bounded lattice* L*, the antiortholattice* $L\oplus L^d$ *is simple.*

Notation 1 *We will denote by* $(\mathrm{B}(L),\leq)$ *the bounded lattice obtained from a lattice* $(L,\leq^L)$ *by adding a new top element* 1 *and a new bottom element* 0*:* $\mathrm{B}(L)=L\amalg\{0\}\amalg\{1\}$ *and* $\leq=\leq^L\cup\{(0,x),(x,1):x\in\mathrm{B}(L)\}\setminus\{(1,0)\}$*.*

Note that, for any lattice L, 0 is meet–irreducible in $\mathrm{B}(L)$; also, 0 is strictly meet–irreducible in $\mathrm{B}(L)$ iff L has a smallest element. Of course, dually for 1. So L is a bounded lattice iff $\mathrm{B}(L)=\mathcal{C}_2\oplus L\oplus\mathcal{C}_2$.

We have $|\mathrm{B}(L)|=|L|+2$ and $\mathrm{Filt}(\mathrm{B}(L))=\{\{1\},\mathrm{B}(L)\}\cup\{F\cup\{1\}:F\in\mathrm{Filt}(L)\}$, $\mathrm{Id}(\mathrm{B}(L))=\{\{0\},\mathrm{B}(L)\}\cup\{I\cup\{0\}:I\in\mathrm{Id}(L)\}$, so that $|\mathrm{Filt}(\mathrm{B}(L))|=|\mathrm{Filt}(L)|+2$ and $|\mathrm{Id}(\mathrm{B}(L))|=|\mathrm{Id}(L)|+2$.

If $(L,\cdot^{\prime L})$ is an involution lattice, then $(\mathrm{B}(L),\cdot')$ is a bounded involution lattice with $\cdot'\mid_L=\cdot^{\prime L}$, $0'=1$ and $1'=0$. Clearly, $(L,\cdot^{\prime L})$ satisfies condition ⓚ iff $(\mathrm{B}(L),\cdot')$ satisfies this condition, case in which $\mathrm{B}(L)$ is a pseudo–Kleene algebra and, furthermore, it becomes an antiortholattice when endowed with the trivial Brouwer complement.

Remark 4 *Let* L *be a lattice and let us keep in mind that, if* $L\in\mathbb{I}$*, then* $\mathrm{B}(L)\in\mathbb{BI}$*.*

Let $\theta\in\mathrm{Con}(L)$ *and* $\alpha\in\mathrm{Con}(\mathrm{B}(L))$*.*

We will consider the equivalence on $\mathrm{B}(L)$ *whose classes are those of* θ *along with the singletons* $\{0\}$ *and* $\{1\}$*:* $\zeta=eq(L/\theta\cup\{\{0\},\{1\}\})$*. Clearly,* $\zeta\cap L^2=\theta$*.*

We will also consider the restriction of α *to* L*:* $\beta=\alpha\cap L^2$*. Note that:* $\alpha=eq(L/\beta\cup\{\{0\},\{1\}\})$ *iff* $\alpha\in\mathrm{Con}_{01}(\mathrm{B}(L))$*.*

From the fact that 0 *is meet–irreducible and* 1 *is join–irreducible in* $\mathrm{B}(L)$ *it is immediate that the equivalence* $eq(\{0\},L,\{1\})$ *on* $\mathrm{B}(L)$ *with singleton classes of* 0 *and* 1 *and all other elements in the same class is a lattice congruence of* $\mathrm{B}(L)$*. Thus* $eq(\{0\},L,\{1\})\in\mathrm{Con}_{01}(\mathrm{B}(L))$*. Hence, if* $L\in\mathbb{I}$*, then* $eq(\{0\},L,\{1\})\in\mathrm{Con}_{\mathbb{BI}01}(\mathrm{B}(L))$*.*

Moreover, $\zeta\in\mathrm{Con}_{01}(\mathrm{B}(L))$*, and, if* $L\in\mathbb{I}$ *and* $\theta\in\mathrm{Con}_{\mathbb{I}}(L)$*, then* $\zeta\in\mathrm{Con}_{\mathbb{BI}01}(\mathrm{B}(L))$*.*

Since L *is a sublattice of* $\mathrm{B}(L)$*, we have* $\beta\in\mathrm{Con}(L)$*. If* $L\in\mathbb{I}$*, then* L *is an i–sublattice of* $\mathrm{B}(L)$*, so, if* $\alpha\in\mathrm{Con}_{\mathbb{I}}(\mathrm{B}(L))$*, then* $\beta\in\mathrm{Con}_{\mathbb{I}}(L)$*.*

Therefore:

- $\mathrm{Con}_{01}(\mathrm{B}(L))=\{eq(L/\gamma\cup\{\{0\},\{1\}\}):\gamma\in\mathrm{Con}(L)\}$*;*

- *if* $L \in \mathbb{I}$*, then* $\mathrm{Con}_{\mathbb{BI}01}(\mathrm{B}(L)) = \{eq(L/\gamma \cup \{\{0\},\{1\}\}) : \gamma \in \mathrm{Con}_{\mathbb{I}}(L)\}$.

See in [14, 9, 15] the congruences of any horizontal sum of nontrivial bounded (involution) lattices. Now let us look at a particular case of horizontal sum: we consider a bounded (involution) lattice L having $|L| > 2$ and the lattice $\mathcal{C}_2^2$, organized as a bi–lattice either as the four–element Boolean algebra or as the horizontal sum $\mathcal{C}_3 \boxplus \mathcal{C}_3$ of two copies of the three–element involution chain. We denote the incomparable elements of $\mathcal{C}_2^2$ by a and b. Let H be the horizontal sum of L with the lattice $\mathcal{C}_2^2$. In the case when L is a bi–lattice, we may organize H as a bi–lattice either as the horizontal sum $L \boxplus \mathcal{C}_2^2$ of L with the four–element Boolean algebra or as the horizontal sum $\mathcal{C}_3 \boxplus L \boxplus \mathcal{C}_3$ of L with two copies of the three–element involution chain; the following hold for any of these two possible definitions of the involution on the lattice $\mathcal{C}_2^2$. See the following diagrams, in which this construction is applied to $\mathrm{B}(M)$ instead of L, for M an (involution) lattice.

Note that $\mathrm{Filt}(H) = (\mathrm{Filt}(L) \setminus \{L\}) \cup \{\{a,1\},\{b,1\},H\}$ and $\mathrm{Id}(H) = (\mathrm{Id}(L) \setminus \{L\}) \cup \{\{0,a\},\{0,b\},H\}$, so $|\mathrm{Filt}(H)| = |\mathrm{Filt}(L)| + 2$ and $|\mathrm{Id}(H)| = |\mathrm{Id}(L)| + 2$.

Clearly, if L satisfies ⓚ, which means that L is a pseudo–Kleene algebra, then the first of these two horizontal sums, namely that of L with the four–element Boolean algebra, satisfies ⓚ, as well, thus, for this definition of the involution, H becomes a pseudo–Kleene algebra.

Remark 5 *With the previous notations, for any element* $x \in L \setminus \{0,1\}$*,* $S = \{0,a,x,b,1\}$ *is a bounded sublattice of* H *and a simple lattice since* $S \cong \mathcal{M}_3$*. Hence, if a lattice congruence* α *of* H *collapses any of the elements* $0,a,x,b,1$*, so that the restriction* $\alpha \cap S^2$ *of* α *to the bounded sublattice* S *of* H *is a nontrivial congruence of* S*, then* $\alpha \cap S^2 = \nabla_S$*, thus* α *collapses* 0 *and* 1 *and thus* $\alpha = \nabla_H$*. Therefore, if* $\alpha \neq \nabla_H$*, then the classes of* $0,a,b,1$ *modulo* α *are singletons, that is* $\alpha \in \mathrm{Con}_{01}(H)$*,* $a/\alpha = \{a\}$ *and* $b/\alpha = \{b\}$*.*

Moreover, for any $\theta \in \mathrm{Con}_{01}(L)$*, the equivalence* $eq(L/\theta \cup \{\{a\},\{b\}\})$*, whose classes are those of* θ *along with the singletons* $\{a\}$ *and* $\{b\}$*, is clearly a lattice congruence of* H*, and it is an i–lattice congruence of* H *if* $\theta \in \mathrm{Con}_{\mathbb{BI}01}(L)$*.*

These properties and the fact that L is a bounded (involution) sublattice of H*, so that, for any (involution) lattice congruence* α *of* H*, the restriction* $\alpha \cap L^2$ *is an (involution) lattice congruence of* L*, prove that:*

- $\mathrm{Con}(H) = \{eq(L/\theta \cup \{\{a\},\{b\}\}) : \theta \in \mathrm{Con}_{01}(L)\} \cup \{\nabla_H\} \cong \mathrm{Con}_{01}(L) \oplus \mathcal{C}_2$*;*

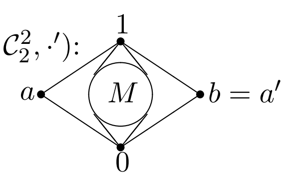
- *if* $L \in \mathbb{BI}$, *so that* $H \in \mathbb{BI}$, *as well, then* $\mathrm{Con}_{\mathbb{I}}(H) = \{eq(L/\theta \cup \{\{a\},\{b\}\}) : \theta \in \mathrm{Con}_{\mathbb{BI}01}(L)\} \cup \{\nabla_H\} \cong \mathrm{Con}_{\mathbb{BI}01}(L) \oplus \mathcal{C}_2$.

Note from the above that, if the bounded lattice L *is* 0*–regular, then* H *is simple as a lattice, thus also as an i–lattice.*

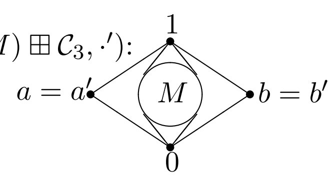
Now let us look at a construction we have used in [4, 14, 15]: let M be a lattice and let us consider the bounded lattice $H = \mathrm{B}(M) \boxplus \mathcal{C}_2^2$, with the incomparable elements of $\mathcal{C}_2^2$ denoted a and b. Then $|H| = |M| + 4$, $\mathrm{Filt}(H) = \{\{1\},\{a,1\},\{b,1\},H\} \cup \{F \cup \{1\} : F \in \mathrm{Filt}(M)\}$ and $\mathrm{Id}(H) = \{\{0\},\{0,a\},\{0,b\},H\} \cup \{I \cup \{0\} : I \in \mathrm{Id}(M)\}$, thus $|\mathrm{Filt}(H)| = |\mathrm{Filt}(M)| + 4$ and $|\mathrm{Id}(H)| = |\mathrm{Id}(M)| + 4$.

If M is an i–lattice, then $\mathrm{B}(M)$ becomes a bi–lattice with the involution that takes 0 to 1 and vice–versa and restricts to the involution of M. As pointed out above, H can be organized as a bi–lattice either as the horizontal sum of bi–lattices $H = \mathrm{B}(M) \boxplus \mathcal{C}_2^2$ of the bi–lattice $\mathrm{B}(M)$ with the four–element Boolean algebra or as the horizontal sum of bi–lattices $H = \mathcal{C}_3 \boxplus \mathrm{B}(M) \boxplus \mathcal{C}_3$ of the bi–lattice $\mathrm{B}(M)$ with two copies of the three–element involution chain:

$(\mathrm{B}(M) \boxplus \mathcal{C}_2^2, \cdot')$:



$(\mathcal{C}_3 \boxplus \mathrm{B}(M) \boxplus \mathcal{C}_3, \cdot')$:

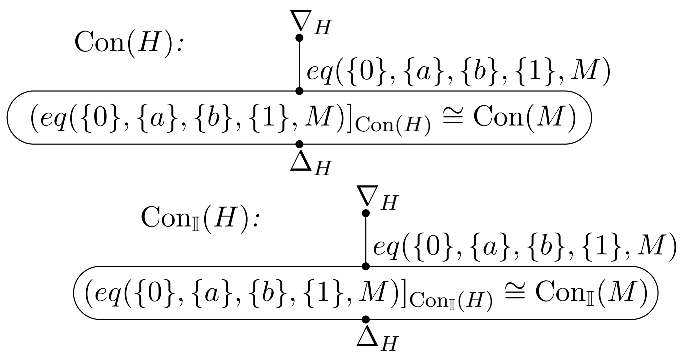


If M is an i–lattice that satisfies ⓚ, then $\mathrm{B}(M)$ is a bi–lattice that satisfies ⓚ, so $\mathrm{B}(M)$ is a pseudo–Kleene algebra. In this case, the horizontal sum $H = \mathrm{B}(M) \boxplus \mathcal{C}_2^2$ of the pseudo–Kleene algebra $\mathrm{B}(M)$ with the four–element Boolean algebra satisfies ⓚ, as well, thus H is a pseudo–Kleene algebra.

Remark 6 *With the previous notations, from the remarks above and the fact that* $\mathrm{B}(M) \setminus \{0,1\} = M$, *we obtain:*

- $\mathrm{Con}(H) = \{eq(M/\theta \cup \{\{0\},\{a\},\{b\},\{1\}\}) : \theta \in \mathrm{Con}(M)\} \cup \{\nabla_H\} \cong \mathrm{Con}(M) \oplus \mathcal{C}_2$, *so* $|\mathrm{Con}(H)| = |\mathrm{Con}(M)| + 1$;

- *if* $M \in \mathbb{I}$, *so that* $H \in \mathbb{BI}$, *then, for either of the definitions above of the involution of* H, *we have* $\mathrm{Con}_{\mathbb{I}}(H) = \{eq(M/\theta \cup \{\{0\},\{a\},\{b\},\{1\}\}) : \theta \in \mathrm{Con}_{\mathbb{I}}(M)\} \cup \{\nabla_H\} \cong \mathrm{Con}_{\mathbb{I}}(M) \oplus \mathcal{C}_2$, *so* $|\mathrm{Con}_{\mathbb{I}}(H)| = |\mathrm{Con}_{\mathbb{I}}(M)| + 1$.



See also [4, 14, 15].

Remark 7 *Let L be a lattice. Since the partitions of L are among the subsets of $\mathcal{P}(L)$ of cardinality at most $|L|$, we have $|\mathrm{Con}(L)| \leq |\mathrm{Eq}(L)| = |\mathrm{Part}(L)| \leq (2^{|L|})^{|L|} = 2^{|L|\cdot|L|}$, so that $|\mathrm{Con}(L)| \leq |\mathrm{Eq}(L)| \leq 2^{|L|}$ if L is infinite. Actually, by [2, 5], if L is finite, then $|\mathrm{Con}(L)| \leq 2^{|L|-1}$, so L has at most as many congruences as subsets regardless of whether it is infinite.*

Concerning the numbers of filters and ideals of L, we have $|L| = |\mathrm{PFilt}(L)| = |\mathrm{PId}(L)| \leq |\mathrm{Filt}(L)|, |\mathrm{Id}(L)| \leq |\mathcal{P}(L)| = 2^{|L|}$. Thus, under the Generalized Continuum Hypothesis, if L is infinite, then $|\mathrm{Filt}(L)|, |\mathrm{Id}(L)| \in \{|L|, 2^{|L|}\}$, while, if, for some infinite cardinal number ν, L has $\{|\mathrm{Filt}(L)|, |\mathrm{Id}(L)|\} = \{\nu, 2^{\nu}\}$, then $|L| = \nu$.

Example 1 *Let ν be an infinite cardinal and C be a well–ordered set with top element having $|C| = \nu$.*

Then C is a bounded chain with all filters principal, thus it has $|\mathrm{Id}(C)| = |\mathrm{Filt}(C)| = |\mathrm{PFilt}(C)| = |C| = \nu$. Hence the Kleene chain $C \oplus C^d$ has $|\mathrm{Filt}(C \oplus C^d)| = |\mathrm{Id}(C \oplus C^d)| = |\mathrm{Filt}(C)| + |\mathrm{Id}(C)| - 1 = \nu + \nu - 1 = \nu$.

For any nonempty subset S of $C \setminus \{0^C, 1^C\}$, if, for each $a \in (\{0^C\} \cup S) \setminus \mathrm{Max}(S)$, a^+ is the successor of a in the well–ordered set $\{0^C\} \cup S$, let us denote by θ_S the equivalence on C whose classes are the nonempty sets $[a, a^+)_C$ for all a as above, along with $\{x \in C : (\forall a \in S)\,(a \geq x)\}$ if this set is nonempty: $\theta_S = eq(\{[a, a^+)_C : a \in (\{0^C\} \cup S) \setminus \mathrm{Max}(S)\} \cup (\{\{x \in C : (\forall a \in S)\,(a \geq x)\}\} \setminus \{\emptyset\}))$.

Since C is a chain, each of its equivalences with all classes convex is a lattice congruence of C [11, 13, 14], hence $\theta_S \in \mathrm{Con}(C)$ for each $S \in \mathcal{P}(C \setminus \{0^C, 1^C\}) \setminus \{\emptyset\}$. Now, if $S, T \in \mathcal{P}(C \setminus \{0^C, 1^C\}) \setminus \{\emptyset\}$ and $S \neq T$, so that, say, there exists an $a \in S \setminus T$, then $[a, a^+)_C \in C/\theta_S \setminus C/\theta_T$, thus $\theta_S \neq \theta_T$.

Hence $S \mapsto \theta_S$ is an injective map from $\mathcal{P}(C \setminus \{0^C, 1^C\}) \setminus \{\emptyset\}$ to $\mathrm{Con}(C)$.

Therefore $2^\nu = 2^{\nu-2} - 1 = |\mathcal{P}(C \setminus \{0^C, 1^C\}) \setminus \{\emptyset\}| \leq |\mathrm{Con}(C)| \leq 2^\nu$, *hence* $|\mathrm{Con}(C)| = 2^\nu$. *Thus* $|\mathrm{Con}(C \oplus C^d)| = |\mathrm{Con}(C)| \cdot |\mathrm{Con}(C)| = 2^\nu \cdot 2^\nu = 2^\nu$.

We may also notice that $C \oplus C^d$ *becomes an antiortholattice when endowed with the trivial Brouwer complement and the following hold.* $\mathrm{Con}_0(C) = \{eq(\{\{0^C\}\} \cup (C \setminus \{0^C\})/\theta) : \theta \in \mathrm{Con}(C \setminus \{0^C\})\} \cong \mathrm{Con}(C \setminus \{0^C\})$. *Since the chain* $C \setminus \{0^C\}$ *is also well ordered and of cardinality* ν, *we have* $|\mathrm{Con}(C \setminus \{0^C\})| = 2^\nu$. *Hence* $|\mathrm{Con}_0(C)| = 2^\nu$, *and, since* $\mathrm{Con}_{\mathbb{BI}01}(C \oplus C^d) = \{\theta \oplus \theta' : \theta \in \mathrm{Con}_0(C)\} \cong \mathrm{Con}_0(C)$, *it follows that* $|\mathrm{Con}_{\mathbb{BI}01}(C \oplus C^d)| = |\mathrm{Con}_0(C)| = 2^\nu$. *Thus the antiortholattice* $C \oplus C^d$ *has* $|\mathrm{Con}_{\mathbb{BZL}}(C \oplus C^d)| = |\mathrm{Con}_{\mathbb{BI}01}(C \oplus C^d) \cup \{\nabla_{C \oplus C^d}\}| = 2^\nu + 1 = 2^\nu$.

Example 2 [4, 14] *Let* ν *be an infinite cardinality and* T *a set with* $|T| = \nu$. *Let us consider the following bounded sublattice of the Boolean algebra* $\mathcal{C}_2^T$: $M = \{(x_t)_{t \in T} \subseteq \mathcal{C}_2 : |\{t \in T : x_t = 1\}| < \aleph_0\} \cup \{1^{\mathcal{C}_2^T}\}$.

Then $|M| = \nu$, *thus* $|M \oplus M^d| = \nu + \nu = \nu$. $M \oplus M^d$ *is a Kleene lattice (with the canonical definition for its involution) and it becomes an antiortholattice when endowed with the trivial Brouwer complement.*

M *satisfies the DCC, thus it has all filters principal, but it has as many ideals as subsets. So* $|M| = |\mathrm{Filt}(M)| = \nu$ *and* $|\mathrm{Id}(M)| = 2^\nu$, *thus* $|\mathrm{Filt}(M \oplus M^d)| = |\mathrm{Id}(M \oplus M^d)| = \nu + 2^\nu - 1 = 2^\nu$.

Since M *is a distributive lattice and thus it has at least as many congruences as ideals, it follows that* M *has as many congruences as subsets:* $|\mathrm{Con}(M)| = 2^\nu$. *Thus* $|\mathrm{Con}(M \oplus M^d)| = |\mathrm{Con}(M)| \cdot |\mathrm{Con}(M)| = 2^\nu \cdot 2^\nu = 2^\nu$ *and* $|\mathrm{Con}_{\mathbb{I}}(M \oplus M^d)| = |\mathrm{Con}(M)| = 2^\nu$.

Note that M *is* 0*–regular. Indeed, let* $\theta \in \mathrm{Con}_0(M)$, *so that* $0^{\mathcal{C}_2^T}/\theta = \{0^{\mathcal{C}_2^T}\}$, *and assume by absurdum that there exist* $x = (x_t)_{t \in T} \in M$ *and* $y = (y_t)_{t \in T} \in M$ *such that* $(x, y) \in \theta$, *but* $x \neq y$. *Then* $x_k \neq y_k$ *for some* $k \in T$; *say* $x_k = 0$ *and* $y_k = 1$. *Let* $z = (z_t)_{t \in T} \in \mathcal{C}_2^T$ *with* $z_k = 1$ *and* $z_t = 0$ *for all* $t \in T \setminus \{k\}$, *so that* $z \in M$. *Then* $y_k \wedge z_k = 1$, *hence* $y \wedge z \neq 0^{\mathcal{C}_2^T}$. *Thus* $(0^{\mathcal{C}_2^T}, y \wedge z) = (x \wedge z, y \wedge z) \in \theta$, *so that* $0^{\mathcal{C}_2^T} \neq y \wedge z \in 0^{\mathcal{C}_2^T}/\theta = \{0^{\mathcal{C}_2^T}\}$, *and we have a contradiction.*

Therefore $\mathrm{Con}_0(M) = \{\Delta_M\}$, *thus* $\mathrm{Con}_{01}(M \oplus M^d) = \{\Delta_{M \oplus M^d}\}$, *so* $\mathrm{Con}_{\mathbb{BI}01}(M \oplus M^d) = \{\Delta_{M \oplus M^d}\}$ *and hence the antiortholattice* $M \oplus M^d$ *is simple:* $\mathrm{Con}_{\mathbb{BZL}}(M \oplus M^d) = \mathrm{Con}_{\mathbb{BI}01}(M \oplus M^d) \cup \{\nabla_{M \oplus M^d}\} = \{\Delta_{M \oplus M^d}, \nabla_{M \oplus M^d}\}$.

5 The Theorems

Throughout the rest of this paper, $\mathbb{V}$ will be an arbitrary variety of algebras of the same similarity type.

Let $(I, \leq)$ be a nonempty ordered set and $(A_\mu)_{\mu \in I}$ a family of members of $\mathbb{V}$. Recall that $(A_\mu)_{\mu \in I}$ is called a *directed system* of members of $\mathbb{V}$ iff it satisfies the following condition, stating that each A_λ is a proper subalgebra of every A_μ with $\lambda < \mu$:

$$\text{Ⓢ}_{\mathbb{V}} \quad \text{for all } \lambda, \mu \in I \text{ with } \lambda < \mu,\ A_\lambda \subseteq A_\mu \text{ and } A_\lambda \in \mathcal{S}_{\mathbb{V}}(A_\mu) \setminus \{A_\mu\}$$

otherwise written:

$$\text{Ⓢ}_{\mathbb{V}} \quad \text{for all } \lambda, \mu \in I \text{ with } \lambda < \mu,\ A_\lambda \subsetneq A_\mu \text{ and } A_\lambda \in \mathcal{S}_{\mathbb{V}}(A_\mu)$$

Of course, if $(A_\mu)_{\mu \in I}$ satisfies condition Ⓢ$_{\mathbb{V}}$, then, for any $\iota \in I$, we have: A_ι is nontrivial iff A_μ is nontrivial for each $\mu \in I$ with $\iota \leq \mu$.

If $(A_\mu)_{\mu \in I}$ is a directed system, then we can define the *directed union* of $(A_\mu)_{\mu \in I}$ to be the member A of $\mathbb{V}$ with $A = \bigcup\limits_{\mu \in I} A_\mu$ and, for every $\star$ belonging to the signature of $\mathbb{V}$ and every $\mu \in I$, $\star^A \mid_{A_\mu} = \star^{A_\mu}$.

Note that, if $(A_\mu)_{\mu \in I}$ is a directed system, then so is $(A_\mu)_{\mu \in J}$ for every nonempty subset J of I, thus, trivially, for each $\mu \in I$, A_μ is the directed union of the family $(A_\lambda)_{\lambda \in I, \lambda \leq \mu}$.

The following condition states that the proper nontrivial congruences of every A_μ are exactly the equivalences on A_μ having an A_λ with $\lambda < \mu$ as unique nonsingleton class:

$$\text{Ⓒ}_{\mathbb{V}} \quad \text{for all } \mu \in I,\ \mathrm{Con}_{\mathbb{V}}(A_\mu) = \{\Delta_{A_\mu}, \nabla_{A_\mu}\} \cup \{eq(\{A_\lambda\} \cup \{\{x\} : x \in A_\mu \setminus A_\lambda\}) : \lambda \in I, \lambda < \mu\}$$

or, equivalently, that the nontrivial congruences of every A_μ are exactly its equivalences having an A_λ with $\lambda \leq \mu$ as unique nonsingleton class:

$$\text{Ⓒ}_{\mathbb{V}} \quad \text{for all } \mu \in I,\ \mathrm{Con}_{\mathbb{V}}(A_\mu) = \{\Delta_{A_\mu}\} \cup \{eq(\{A_\lambda\} \cup \{\{x\} : x \in A_\mu \setminus A_\lambda\}) : \lambda \in I, \lambda \leq \mu\}$$

Note that a singleton family satisfies condition Ⓒ$_{\mathbb{V}}$ iff its member is a simple algebra from $\mathbb{V}$.

If $\mathbb{V}$ is the variety of lattices or that of bounded lattices, then we denote the conditions Ⓢ$_{\mathbb{V}}$ and Ⓒ$_{\mathbb{V}}$, simply, by Ⓢ and Ⓒ, respectively.

Before proceeding towards the main results, let us take a look at the structures that the poset $(I, \leq)$ and the family $(A_\mu)_{\mu \in I}$ ordered by set inclusion need to have so that conditions Ⓢ$_{\mathbb{V}}$ and ⓒ$_{\mathbb{V}}$ can be satisfied.

Remark 8 *Let $(I, \leq)$ be a nonempty ordered set and $(A_\mu)_{\mu \in I}$ be a directed system of nontrivial members of $\mathbb{V}$ which satisfies conditions Ⓢ$_{\mathbb{V}}$ and ⓒ$_{\mathbb{V}}$. Then:*

- *the map $\mu \mapsto A_\mu$ from the poset $(I, \leq)$ to the poset $((A_\mu)_{\mu \in I}, \subseteq)$ is an order–preserving bijection;*
- *if $(I, \leq)$ is a chain, then so is $((A_\mu)_{\mu \in I}, \subseteq)$, and the map $\mu \mapsto A_\mu$ from $(I, \leq)$ to $((A_\mu)_{\mu \in I}, \subseteq)$ is a lattice isomorphism.*

Indeed, the map $\mu \mapsto A_\mu$ from I to $(A_\mu)_{\mu \in I}$ is clearly surjective; by condition Ⓢ$_{\mathbb{V}}$, it is also order–preserving and injective; thus it is an order–preserving bijection between these two posets. Hence the statement for the particular case when $(I, \leq)$ is a chain.

Now let $\mu \in I$ and let us denote by $(\mu]_I = \{\lambda \in I : \lambda \leq \mu\}$. Let us consider the restriction of the map above to the order ideal $(\mu]_I$, and also consider the map $A_\lambda \mapsto eq(\{A_\lambda\} \cup \{\{x\} : x \in A_\mu \setminus A_\lambda\})$ defined on $(A_\lambda)_{\lambda \in (\mu]_I}$. The latter map is clearly injective and order–preserving, and its image does not contain Δ_{A_μ} since each algebra A_λ is nontrivial, hence, by condition ⓒ$_{\mathbb{V}}$, the image of this map is $\mathrm{Con}_{\mathbb{V}}(A_\mu) \setminus \{\Delta_{A_\mu}\}$. It follows that:

- *the map $\lambda \mapsto A_\lambda$ from $((\mu]_I, \leq)$ to $((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$ and the map $A_\lambda \mapsto eq(\{A_\lambda\} \cup \{\{x\} : x \in A_\mu \setminus A_\lambda\})$ from $((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$ to $(\mathrm{Con}_{\mathbb{V}}(A_\mu) \setminus \{\Delta_{A_\mu}\}, \subseteq)$ are order–preserving bijections, thus so is their composition, namely the map $\lambda \mapsto eq(\{A_\lambda\} \cup \{\{x\} : x \in A_\mu \setminus A_\lambda\})$ from $(\mu]_I$ to $(\mathrm{Con}_{\mathbb{V}}(A_\mu) \setminus \{\Delta_{A_\mu}\}, \subseteq)$; in particular, $|\mathrm{Con}_{\mathbb{V}}(A_\mu)| = |(\mu]_I| + 1$;*
- *thus: $((\mu]_I, \leq)$ has a least element iff $((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$ has a least element iff $(\mathrm{Con}_{\mathbb{V}}(A_\mu) \setminus \{\Delta_{A_\mu}\}, \subseteq)$ has a least element iff $\mathrm{Con}_{\mathbb{V}}(A_\mu)$ has a single atom iff A_μ is subdirectly irreducible;*
- *if $((\mu]_I, \leq)$ is a chain, then so are $((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$ and $(\mathrm{Con}_{\mathbb{V}}(A_\mu) \setminus \{\Delta_{A_\mu}\}, \subseteq)$, thus the maps above are lattice isomorphisms;*
- *thus, if $((\mu]_I, \leq)$ is a chain and it has a least element, so that it is a bounded chain, then $((\mu]_I, \leq)$, $((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$ and $\mathrm{Con}_{\mathbb{V}}(A_\mu)$ are complete chains such that $((\mu]_I, \leq) \cong ((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$ and $\mathrm{Con}_{\mathbb{V}}(A_\mu) \cong \mathcal{C}_2 \oplus ((\mu]_I, \leq) \cong \mathcal{C}_2 \oplus ((A_\lambda)_{\lambda \in (\mu]_I}, \subseteq)$.*

The next two lemmas offer a method to construct, in any variety $\mathbb{V}$, a family of algebras with any numbers of congruences, provided for every successor ordinal $\mu+1$ with $\mu > 0$ and any family $(A_\lambda)_{2\leq\lambda\leq\mu}$ of members of $\mathbb{V}$ satisfying conditions Ⓢ$_\mathbb{V}$ and Ⓒ$_\mathbb{V}$, we can construct an algebra $A_{\mu+1}$ from $\mathbb{V}$ such that the family $(A_\lambda)_{2\leq\lambda\leq\mu+1}$ also satisfies conditions Ⓢ$_\mathbb{V}$ and Ⓒ$_\mathbb{V}$: that is, an algebra $A_{\mu+1}$ from $\mathbb{V}$ which includes every member of the family $(A_\lambda)_{2\leq\lambda\leq\mu}$ and whose nontrivial congruences are exactly its equivalences having one of the members of $(A_\lambda)_{2\leq\lambda\leq\mu+1}$ as unique nonsingleton class. Additionally, if we let A_2 be infinite, of a cardinality ν, and the construction lets $|A_{\mu+1}|$ differ from $|A_\mu|$ by a cardinality less than ν, so that we have, in fact, $|A_{\mu+1}| = |A_\mu|$, then, for any ordinal τ with $|\tau| \leq \nu$, we can obtain a family $(A_\lambda)_{2\leq\lambda\leq\tau}$ of algebras of cardinality ν whose numbers of congruences take every value between 2 and $|\tau|$.

Lemma 1 [4, Lemma 3.2] *Let ι be a limit ordinal, σ an ordinal with $\sigma < \iota$, $I = \{\mu : \sigma \leq \mu < \iota\}$, $(A_\mu)_{\mu\in I}$ a family of members of $\mathbb{V}$ and A_ι the directed union of the family $(A_\mu)_{\mu\in I}$. If the family $(A_\mu)_{\mu\in I}$ satisfies conditions Ⓢ$_\mathbb{V}$ and Ⓒ$_\mathbb{V}$, then the family $(A_\mu)_{\mu\in I\cup\{\iota\}}$ also satisfies conditions Ⓢ$_\mathbb{V}$ and Ⓒ$_\mathbb{V}$.*

Lemma 2 *Let τ be an ordinal, $I = \{\mu : 2 \leq \mu \leq \tau\}$ and $(A_\mu)_{\mu\in I}$ be a family of nontrivial members of $\mathbb{V}$ that satisfies conditions Ⓢ$_\mathbb{V}$ and Ⓒ$_\mathbb{V}$.*

(i) Then, for all $\mu \in I$, $\mathrm{Con}_\mathbb{V}(A_\mu)$ is isomorphic to the chain $\{\lambda : 1 \leq \lambda \leq \mu\}$, in particular $\mathrm{Con}_\mathbb{V}(A_\mu)$ is a well–ordered set with $|\mathrm{Con}_\mathbb{V}(A_\mu)| = |\mu|$.

(ii) Assume that A_2 is infinite and has $|A_2| = \nu \geq |\tau|$, that, for each ordinal $\mu \in I$ such that $\mu + 1 \in I$, we have $|A_{\mu+1}| = |A_\mu|$, and that, for each limit ordinal $\iota \in I$, A_ι is the directed union of the family $(A_\lambda)_{\lambda\in I,\lambda<\iota}$. Then $|A_\lambda| = \nu$ for each $\lambda \in I$.

Proof: (i) Let $\mu \in I$. By Remark 8, since $I = \{\lambda : 2 \leq \lambda \leq \tau\}$ is a chain with least element 2 and thus $\{\lambda : 2 \leq \lambda \leq \mu\}$ is a chain with least element 2, we have $\mathrm{Con}_\mathbb{V}(A_\mu) \cong \mathcal{C}_2 \oplus \{\lambda : 2 \leq \lambda \leq \mu\} \cong \{\lambda : 1 \leq \lambda \leq \mu\}$.
(ii) We apply induction. By the hypothesis, $|A_2| = \nu$.

Now let $\iota \in I \setminus \{2\}$, which means that ι is an ordinal with $3 \leq \iota \leq \tau$.

If ι is a successor ordinal, $\iota = \mu + 1$ for a (unique) ordinal μ with $2 \leq \mu < \tau$ and such that $|A_\mu| = \nu$, then $|A_\iota| = |A_\mu| = \nu$ by the assumption in the enunciation of (ii).

If ι is a limit ordinal such that, for all ordinals μ with $2 \leq \mu < \iota$, $|A_\mu| = \nu$, then $\nu = |A_2| \leq |A_\iota| \leq \sum_{2\leq\lambda<\iota} |A_\mu| = \sum_{2\leq\lambda<\iota} \nu \leq |\iota| \cdot \nu \leq |\tau| \cdot \nu = \nu$ since $\nu \geq |\tau|$. □

Proposition 1 *Let $\kappa \geq 3$ be a cardinal number, ι the smallest ordinal with $|\iota| = \kappa$ and $I = \{\mu : 2 \leq \mu < \iota\}$.*

If there exists a family $(A_\mu)_{\mu\in I}$ of nontrivial members of $\mathbb{V}$ that satisfies conditions ⓢ$_{\mathbb{V}}$ and ⓒ$_{\mathbb{V}}$, then, for any nonsingleton well–ordered set $(S, \leq)$ having a largest element and $|S| < \kappa$, there exists a member A of the family $(A_\mu)_{\mu\in I}$ such that $\mathrm{Con}_{\mathbb{V}}(A)$ is isomorphic to $(S, \leq)$.

Proof: Let 1^S be the largest element of $(S, \leq)$ and let us denote by $Sji(S)$ the set of the strictly join–irreducible elements of the bounded chain $(S, \leq)$, that is the elements of this chain that have an (obviously unique) predecessor. Of course, if S is finite, then $Sji(S) = S \setminus \{\min(S, \leq)\}$.

Let σ and τ be ordinals with $|\sigma| = |\tau| = |S|$, such that τ is a successor ordinal and σ is a limit ordinal. Then $\tau \in I$ and, provided σ exists (which, of course, is not the case if S is finite, case in which $1^S \in Sji(S)$), we also have $\sigma \in I$.

By Lemma 2, (i), a member A of the family $(A_\mu)_{\mu\in I}$ whose congruence lattice is isomorphic to $(S, \leq)$ is: $A = \begin{cases} A_\sigma, & \text{if } 1^S \notin Sji(S), \\ A_\tau, & \text{if } 1^S \in Sji(S). \end{cases}$ □

We will use the general method offered by Lemma 2 in Theorem 2 below, much in the same way as it has been used in [4] to obtain the next theorem; we revisit this method to obtain the cases that do not follow directly from this Theorem 1 as described in Remark 10, namely the case in which our i–lattices have as many ideals as elements, along with the cases in which they have as many congruences as their lattice reducts; but first we are going to point out that, when applied to an infinite simple lattice A_2, our construction produces a family of lattices each with as many elements, ideals and filters as A_2.

Lemma 3 [4, Lemma 3.1] *Let I be an ideal of a lattice K such that K satisfies the following condition:*

ⓖ$_I$ *for all $(x_n)_{n\in\mathbb{N}} \subseteq K$, if $x_n > x_{n+1}$ for all $n \in \mathbb{N}$, then $x_n \in I$ for all but finitely many $n \in \mathbb{N}$.*

Then every nonprincipal filter of K is generated by a filter of I.

Theorem 1 [4, Theorem 1.1] *For any infinite cardinal number ν and any cardinal number κ with $2 \leq \kappa \leq \nu$ or $\kappa = 2^\nu$, there exists a bounded lattice $M_{\nu,\kappa}$ with $|M_{\nu,\kappa}| = |\mathrm{Filt}(M_{\nu,\kappa})| = \nu$, $|\mathrm{Id}(M_{\nu,\kappa})| = 2^\nu$ and $|\mathrm{Con}(M_{\nu,\kappa})| = \kappa$. Furthermore, $M_{\nu,2^\nu}$ can be chosen to be distributive.*

Under the Generalized Continuum Hypothesis, with ν, κ and the notations as in the previous theorem, the lattices $M_{\nu,\kappa}$ and $M^d_{\nu,\kappa}$ have the only possible values for the cardinalities of their sets of filters and ideals under the condition that these cardinalities are different.

Remark 9 *With ν, κ and the notations from Theorem 1, if $\kappa \leq \nu$, then, additionally, for any successor ordinal τ with $|\tau| = \kappa$, $M_{\nu,\kappa}$ can be chosen such that $\mathrm{Con}(M_{\nu,\kappa})$ is isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \tau\}$.*

On the other hand, for every limit ordinal σ with $|\sigma| = \kappa \leq \nu$, there exists a lattice $M_{\nu,\kappa}$ without lattice bounds having $|M_{\nu,\kappa}| = |\mathrm{Filt}(M_{\nu,\kappa})| = \nu$, $|\mathrm{Id}(M_{\nu,\kappa})| = 2^\nu$ and $\mathrm{Con}(M_{\nu,\kappa})$ isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \sigma\}$, in particular having $|\mathrm{Con}(M_{\nu,\kappa})| = \kappa$.

Remark 10 *Let ν be an infinite cardinal number. For every cardinal number κ with $2 \leq \kappa \leq \nu$ or $\kappa = 2^\nu$, with the notations from Theorem 1, we let $L_{\nu,\kappa} = M_{\nu,\kappa} \oplus M^d_{\nu,\kappa}$. Then $L_{\nu,\kappa}$ is a pseudo–Kleene algebra and, by Theorem 1, $L_{\nu,2^\nu}$ can be chosen to be a Kleene lattice.*

We have $|\mathrm{Filt}(L_{\nu,\kappa})| = |\mathrm{Id}(L_{\nu,\kappa})| = |\mathrm{Filt}(M_{\nu,\kappa})| + |\mathrm{Id}(M_{\nu,\kappa})| - 1 = \nu + 2^\nu - 1 = 2^\nu$.

$\mathrm{Con}(L_{\nu,\kappa}) \cong \mathrm{Con}(M_{\nu,\kappa})^2$ and $\mathrm{Con}_{\mathbb{I}}(L_{\nu,\kappa}) \cong \mathrm{Con}(M_{\nu,\kappa})$, thus $|\mathrm{Con}(L_{\nu,\kappa})| = \kappa^2$ and $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,\kappa})| = \kappa$, in particular $\mathrm{Con}(L_{\nu,\kappa}) \cong \mathrm{Con}_{\mathbb{I}}(L_{\nu,\kappa})^2$ and, if κ is infinite, then $|\mathrm{Con}(L_{\nu,\kappa})| = |\mathrm{Con}_{\mathbb{I}}(L_{\nu,\kappa})|$.

Under the Generalized Continuum Hypothesis, the cardinal numbers κ above take each value between 2 and the cardinality 2^ν of the sets of the subsets of the lattices $L_{\nu,\kappa}$.

Also, by the previous remark, for each such cardinal κ and every successor ordinal τ with $|\tau| = \kappa$, $M_{\nu,\kappa}$ can be chosen such that $\mathrm{Con}_{\mathbb{I}}(L_{\nu,\kappa}) \cong \mathrm{Con}(M_{\nu,\kappa}) \cong \{\lambda : 1 \leq \lambda \leq \tau\}$.

Now we revisit the technique from the proof in [4] of Theorem 1; we apply the construction from [4, Section 3] to an arbitrary lattice L and we also consider the case when L is an involution lattice. We will apply Lemma 3 in a slightly different manner than in [4, Section 3], so that, in statement (v) of the following proposition, we do not need to confine ourselves to the cases when L satisfies the DCC or the ACC or has as many filters or ideals as subsets; instead, this statement holds for any lattice L and does not necessitate enforcing the Continuum Hypothesis.

Remark 11 *Of course, if a lattice L has all filters principal, then $|\mathrm{Filt}(L)| = |L|$, and the same goes for ideals, but the converses do not hold; for instance, if ν is an infinite cardinal number and we let $L = \mathcal{M}_{2^\nu} \oplus \mathcal{C}_2^\nu$, then, since the Boolean algebra $\mathcal{C}_2^\nu$ has as many filters and as many ideals and subsets, while the modular lattice $\mathcal{M}_{2^\nu}$ has finite length and thus all filters and ideals principal, we have $|\mathrm{Filt}(L)| = |\mathrm{Id}(L)| = |\mathrm{Filt}(\mathcal{M}_{2^\nu})| + |\mathrm{Filt}(\mathcal{C}_2^\nu)| - 1 = |\mathrm{Id}(\mathcal{M}_{2^\nu})| + |\mathrm{Id}(\mathcal{C}_2^\nu)| - 1 = 2^\nu + 2^\nu - 1 = 2^\nu = |L|$, and L has nonprincipal filters, namely the nonprincipal filters of $\mathcal{C}_2^\nu$, and nonprincipal ideals, namely the unions of $\mathcal{M}_{2^\nu}$ with nonprincipal ideals of $\mathcal{C}_2^\nu$; see in [13] more examples of lattices with as many filters and ideals as elements, but having nonprincipal filters and nonprincipal ideals, thus failing both the DCC and the ACC.*

Now let us apply the construction above to an algebra $A_2 = L$ that can be a lattice or an i–lattice. So let L be an (involution) lattice and κ be a cardinal number with $2 \leq \kappa$; let σ be an ordinal with $|\sigma| = \kappa$ and $\tau = \begin{cases} \sigma, & \text{if } \kappa \text{ is finite,} \\ \sigma + 1, & \text{if } \kappa \text{ is infinite} \end{cases}$, so that τ is a successor ordinal with $\sigma \leq \tau$ and $|\tau| = \kappa$; and let $I = \{\mu : 2 \leq \mu \leq \tau\}$.

We define inductively a family $(L_\mu)_{\mu\in I}$ of (involution) lattices, in the following way: $L_2 = L$ and, for every $\iota \in I \setminus \{2\} = \{\mu : 3 \leq \mu \leq \tau\}$:

- if ι is a successor ordinal, $\iota = \mu + 1$ for a $\mu \in I$, then we define $L_\iota = \mathrm{B}(L_\mu) \boxplus \mathcal{C}_2^2$, as a horizontal sum of bounded (involution) lattices;
- if ι is a limit ordinal, then we define the (involution) lattice L_ι to be the directed union of the family of (involution) lattices $(L_\mu)_{2\leq\mu<\iota}$.

In the next remarks and proposition, we keep the notations above, namely the cardinal number $\kappa \geq 2$, the ordinal σ having the cardinality κ, the successor ordinal $\tau \geq \sigma$ also having the cardinality κ, the index set I of the ordinals between 2 and τ, endowed with the canonical good order, the (involution) lattice L and the family $(L_\mu)_{\mu\in I}$ of (involution) lattices defined as above. We will always consider the canonical order on any set of ordinals.

Remark 12 *Note that, in the subfamily $(L_\mu)_{\mu\in I\setminus\{2\}}$ of (involution) lattices, the bounded members L_μ are exactly those indexed by successor ordinals μ. In particular, L_τ is a bounded (involution) lattice.*

Remark 13 *For the case of bounded involution lattices in the following proposition, we can also define, for every $\mu \in I$ such that $\iota = \mu + 1 \in I$, the bi–lattice L_ι to be the horizontal sum of bi–lattices $\mathcal{C}_3 \boxplus \mathrm{B}(L_\mu) \boxplus \mathcal{C}_3$, and the statements in the proposition still hold.*

However, if we let the involution of L_ι to be defined as in the horizontal sum of the bi–lattice $\mathrm{B}(L_\mu)$ with the four–element Boolean algebra, and L satisfies condition ⓚ (so that L is a pseudo–Kleene algebra in the particular case when L is bounded), then it is easy to see that each member of the family $(L_\mu)_{\mu \in I}$ satisfies ⓚ, and thus L_ι is a pseudo–Kleene algebra for every successor ordinal $\iota \in I \setminus \{2\}$, in particular L_τ is a pseudo–Kleene algebra.

Proposition 2 *With the notations above:*

(i) *the family $(L_\mu)_{\mu \in I}$ satisfies condition ⓢ and, if $L \in \mathbb{I}$, so that $(L_\mu)_{\mu \in I} \subset \mathbb{I}$, then also condition ⓢ$_{\mathbb{I}}$;*

(ii) *if L is a nontrivial simple lattice, then the family $(L_\mu)_{\mu \in I}$ satisfies condition ⓒ, so, for all $\mu \in I$, $\mathrm{Con}(L_\mu)$ is isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \mu\}$ and thus $|\mathrm{Con}(L_\mu)| = |\mu|$, in particular $\mathrm{Con}(L_\sigma)$ and $\mathrm{Con}(L_\tau)$ are isomorphic to the well–ordered sets $\{\lambda : 1 \leq \lambda \leq \sigma\}$ and $\{\lambda : 1 \leq \lambda \leq \tau\}$, respectively, and thus $|\mathrm{Con}(L_\sigma)| = |\mathrm{Con}(L_\tau)| = \kappa$;*

(iii) *if L is a nontrivial simple i–lattice, then the family $(L_\mu)_{\mu \in I}$ satisfies condition ⓒ$_{\mathbb{I}}$, so, for all $\mu \in I$, $\mathrm{Con}_{\mathbb{I}}(L_\mu)$ is isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \mu\}$ and thus $|\mathrm{Con}_{\mathbb{I}}(L_\mu)| = |\mu|$, in particular $\mathrm{Con}_{\mathbb{I}}(L_\sigma)$ and $\mathrm{Con}_{\mathbb{I}}(L_\tau)$ are isomorphic to the well–ordered sets $\{\lambda : 1 \leq \lambda \leq \sigma\}$ and $\{\lambda : 1 \leq \lambda \leq \tau\}$, respectively, and thus $|\mathrm{Con}_{\mathbb{I}}(L_\sigma)| = |\mathrm{Con}_{\mathbb{I}}(L_\tau)| = \kappa$;*

(iv) *if L is a nontrivial i–lattice with a simple lattice reduct, then the family $(L_\mu)_{\mu \in I}$ satisfies conditions ⓒ and ⓒ$_{\mathbb{I}}$, so, for all $\mu \in I$, $\mathrm{Con}_{\mathbb{I}}(L_\mu) = \mathrm{Con}(L_\mu) \cong \{\lambda : 1 \leq \lambda \leq \mu\}$ and thus $|\mathrm{Con}_{\mathbb{I}}(L_\mu)| = |\mathrm{Con}(L_\mu)| = |\mu|$, in particular $\mathrm{Con}_{\mathbb{I}}(L_\sigma) = \mathrm{Con}(L_\sigma) \cong \{\lambda : 1 \leq \lambda \leq \sigma\}$, $\mathrm{Con}_{\mathbb{I}}(L_\tau) = \mathrm{Con}(L_\tau) \cong \{\lambda : 1 \leq \lambda \leq \tau\}$ and thus $|\mathrm{Con}_{\mathbb{I}}(L_\sigma)| = |\mathrm{Con}(L_\sigma)| = |\mathrm{Con}_{\mathbb{I}}(L_\tau)| = |\mathrm{Con}(L_\tau)| = \kappa$;*

(v) *if L is an infinite lattice and $|L| = \nu \geq \kappa$, then, for all $\mu \in I$, $|L_\mu| = \nu$, $|\mathrm{Filt}(L_\mu)| = |\mathrm{Filt}(L)|$ and $|\mathrm{Id}(L_\mu)| = |\mathrm{Id}(L)|$, in particular $|L_\sigma| = |L_\tau| = \nu$, $|\mathrm{Filt}(L_\sigma)| = |\mathrm{Filt}(L_\tau)| = |\mathrm{Filt}(L)|$ and $|\mathrm{Id}(L_\sigma)| = |\mathrm{Id}(L_\tau)| = |\mathrm{Id}(L)|$.*

Proof: (i) The singleton family $\{L_2\} = \{L\}$ trivially satisfies condition ⓢ, respectively $ⓢ_{\mathbb{I}}$. For every $\iota \in I \setminus \{1\}$, if ι is a successor ordinal, $\iota = \mu + 1$ for some $\mu \in I$, then, by the definition of L_ι, we have $L_\mu \in \mathcal{S}(L_\iota)$, respectively $L_\mu \in \mathcal{S}_{\mathbb{I}}(L_\iota)$, while, if ι is a limit ordinal, then, again by the definition of L_ι, we have $L_\lambda \in \mathcal{S}(L_\iota)$, respectively $L_\lambda \in \mathcal{S}_{\mathbb{I}}(L_\iota)$, for each $2 \leq \lambda < \iota$. So an immediate induction argument shows that the family $(L_\mu)_{\mu \in I}$ satisfies condition ⓢ, respectively $ⓢ_{\mathbb{I}}$.

(ii),(iii) We apply induction. Assume that $L_2 = L$ is a simple lattice, respectively a simple i–lattice, so that the singleton family $\{L_2\} = \{L\}$ satisfies condition ⓒ, respectively $ⓒ_{\mathbb{I}}$. Now let $\iota \in I \setminus \{2\}$, and let $\mathbb{V}$ be the variety of lattices in the case of (ii), respectively $\mathbb{V} = \mathbb{I}$ in the case of (iii).

If ι is a successor ordinal, $\iota = \mu + 1$ for some $\mu \in I$ such that the family $(L_\lambda)_{2 \leq \lambda \leq \mu}$ satisfies condition $ⓒ_{\mathbb{V}}$, then $L_\iota = \mathrm{B}(L_\mu) \boxplus \mathcal{C}_2^2$ and $\mathrm{Con}_{\mathbb{V}}(L_\mu) = \{\Delta_{L_\mu}\} \cup \{eq(\{L_\lambda\} \cup \{\{x\} : x \in L_\mu \setminus L_\lambda\}) : 2 \leq \lambda \leq \mu\}$, therefore, by (i) and the congruences of the construction $L_\iota = \mathrm{B}(L_\mu) \boxplus \mathcal{C}_2^2$ determined in Remark 6, $\mathrm{Con}_{\mathbb{V}}(L_\iota) = \{\nabla_{L_\iota}\} \cup \{eq(L_\mu/\theta \cup \{\{x\} : x \in \mathcal{C}_2^2 = L_\iota \setminus L_\mu\}) : \theta \in \mathrm{Con}_{\mathbb{V}}(L_\mu)\} = \{\Delta_{L_\iota}, \nabla_{L_\iota}\} \cup \{eq(\{L_\lambda\} \cup \{\{x\} : x \in L_\iota \setminus L_\lambda\}) : 2 \leq \lambda \leq \mu\}$, hence the family $(L_\lambda)_{2 \leq \lambda \leq \iota}$ also satisfies condition $ⓒ_{\mathbb{V}}$.

If ι is a limit ordinal such that the family $(L_\lambda)_{2 \leq \lambda < \iota}$ satisfies condition $ⓒ_{\mathbb{V}}$, then, by (i) and Lemma 1, the family $(L_\lambda)_{2 \leq \lambda \leq \iota}$ also satisfies condition $ⓒ_{\mathbb{V}}$.

By the transfinite induction principle, it follows that the family $(L_\mu)_{2 \leq \mu \leq \tau}$ satisfies condition $ⓒ_{\mathbb{V}}$. By (i) and Lemma 2, (i), it follows that, for all ordinals μ with $2 \leq \mu \leq \tau$, $\mathrm{Con}_{\mathbb{V}}(L_\mu)$ is isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \mu\}$, so $|\mathrm{Con}_{\mathbb{V}}(L_\mu)| = |\mu|$.

(iv) By (ii) and (iii), along with the description of the congruences in conditions ⓒ and $ⓒ_{\mathbb{I}}$ and the obvious fact that, if the lattice reduct of the i–lattice L is simple, then so is the i–lattice L.

(v) By Lemma 2, (ii), we have $|L_\mu| = \nu$ for all $\mu \in I$. The property of the numbers of filters and ideals is trivial for $L_2 = L$.

Now let $\iota \in I \setminus \{2\}$. Let us consider the ideal $J = (L]_{L_\iota} = (1^{L_3}]_{L_\iota} \setminus \{1^{L_3}\} = (0^{L_3}]_{L_\iota} \cup L = \{0^{L_{\lambda+1}} : 2 \leq \lambda < \tau\} \cup L$ since the chain $(0^{L_3}]_{L_\iota}$ is formed of the elements 0^{L_μ} with μ a successor ordinal in I if $L_2 = L$ is bounded and in $I \setminus \{2\}$ otherwise. J is a principal ideal of L_ι iff L has a top element. Since the set $\{\mu : 2 \leq \mu \leq \iota\}$ is well ordered and thus so is the filter $[1^{L_3})_{L_\iota} = \{1^{L_{\lambda+1}} : 2 \leq \lambda < \iota\}$, it is easy to notice that L_ι satisfies the property $ⓖ_J$, so, by Lemma 3, every nonprincipal filter of L_ι is generated by a filter of J.

Let F be a nonprincipal filter of L_ι. Then there exists a filter G of $J = (0^{L_3}]_{L_\iota} \cup L$ such that $F = [G)_{L_\iota}$.

If $G \subseteq L$, so that G is a filter of L, then $F = [G)_{L_\iota} = G \cup [1^{L_3})_{L_\iota}$, hence G is nonprincipal since F is nonprincipal.

If $G \nsubseteq L$, then $G \cap (J \setminus L) = G \cap (0^{L_3}]_{L_\iota}$ is nonempty, hence $H = G \cap (0^{L_3}]_{L_\iota}$ is a filter of $(0^{L_3}]_{L_\iota}$ and clearly, if we denote by a_{L_μ}, b_{L_μ} the two incomparable elements of the copy of $\mathcal{C}_2^2$ from $L_\mu = \mathrm{B}(L_\lambda) \boxplus \mathcal{C}_2^2$ for each successor ordinal $\mu = \lambda + 1$ with $\lambda \in I \setminus \{\tau\}$, then $F = [G)_{L_\iota} = [H)_{L_\iota} = H \cup \bigcup_{\mu \in I, 0^{L_\mu} \in H} [0^{L_\mu}, 1^{L_\mu}]_{L_\iota} \cup L \cup [1^{L_3})_{L_\iota} = H \cup \{0^{L_\mu}, a_{L_\mu}, b_{L_\mu} : \mu \in I, 0^{L_\mu} \in H\} \cup L \cup [1^{L_3})_{L_\iota}$, and thus H is nonprincipal since F is nonprincipal. $(0^{L_3}]_{L_\iota} = \{\lambda + 1 : 2 \leq \lambda < \iota\}$ is dually well ordered, hence it has all ideals principal and thus, since it is a chain, $|\mathrm{Filt}((0^{L_3}]_{L_\iota})| = |\mathrm{Id}((0^{L_3}]_{L_\iota})| = |(0^{L_3}]_{L_\iota}| \leq |\iota| \leq |\tau| = \kappa \leq \nu$.

By the above, clearly, G (thus also $G \cap (0^{L_3}]_{L_\iota}$ in the second case above) is uniquely determined by F, and hence $|\mathrm{Filt}(L_\iota)| = |\mathrm{PFilt}(L_\iota)| + |\mathrm{Filt}(L_\iota) \setminus \mathrm{PFilt}(L_\iota)| = |L_\iota| + |\mathrm{Filt}(J) \setminus \mathrm{PFilt}(J)| = \nu + |\mathrm{Filt}(J) \setminus \mathrm{PFilt}(J)| = |L| + |\mathrm{Filt}(L) \setminus \mathrm{PFilt}(L)| + |\mathrm{Filt}((0^{L_3}]_{L_\iota}) \setminus \mathrm{PFilt}((0^{L_3}]_{L_\iota})| = |\mathrm{PFilt}(L)| + |\mathrm{Filt}(L) \setminus \mathrm{PFilt}(L)| + |\mathrm{Filt}((0^{L_3}]_{L_\iota}) \setminus \mathrm{PFilt}((0^{L_3}]_{L_\iota})| = |\mathrm{Filt}(L)| + |\mathrm{Filt}((0^{L_3}]_{L_\iota}) \setminus \mathrm{PFilt}((0^{L_3}]_{L_\iota})| = |\mathrm{Filt}(L)|$ since $|\mathrm{Filt}(L)| \geq |\mathrm{PFilt}(L)| = |L| = \nu \geq |\mathrm{Filt}((0^{L_3}]_{L_\iota})| \geq |\mathrm{Filt}((0^{L_3}]_{L_\iota}) \setminus \mathrm{PFilt}((0^{L_3}]_{L_\iota})|$.

By duality, it follows that $|\mathrm{Id}(L_\iota)| = |\mathrm{Id}(L)|$. □

Corollary 1 *For any infinite simple (involution) lattice L, every cardinal number κ with $2 \leq \kappa \leq |L|$ and every ordinal ι with $|\iota| = \kappa$, there exists an (involution) lattice M with $|M| = |L|$, $|\mathrm{Filt}(M)| = |\mathrm{Filt}(L)|$, $|\mathrm{Id}(M)| = |\mathrm{Id}(L)|$ and:*

(i) if L is a simple lattice, then $\mathrm{Con}(M)$ is isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \iota\}$, in particular $|\mathrm{Con}(M)| = \kappa$;

(ii) if L is a simple i–lattice, then $\mathrm{Con}_{\mathbb{I}}(M)$ is isomorphic to the well–ordered set $\{\lambda : 1 \leq \lambda \leq \iota\}$, in particular $|\mathrm{Con}_{\mathbb{I}}(M)| = \kappa$;

(iii) if L is an i–lattice with a simple lattice reduct, then $\mathrm{Con}_{\mathbb{I}}(M) = \mathrm{Con}(M) \cong \{\lambda : 1 \leq \lambda \leq \iota\}$, in particular $|\mathrm{Con}_{\mathbb{I}}(M)| = |\mathrm{Con}(M)| = \kappa$;

(iv) if ι is a successor ordinal, then M is a bounded (involution) lattice;

(v) if L is an i–lattice and satisfies condition Ⓚ*, then the i–lattice M satisfies condition* Ⓚ*, thus M is a pseudo–Kleene algebra if, additionally, ι is a successor ordinal.*

Proof: We apply to L the construction above, with $L_2 = L$ and $\iota = \sigma$, and take $M = L_\iota$. Then we apply Proposition 2 to obtain (i), (ii) and (iii), then take $\iota = \tau$ to also get (iv). Finally, we apply Remark 13 to obtain (v). □

Obviously, we can obtain congruence lattices with different shapes without changing these cardinalities, at least in the infinite case. For instance, given any i–lattice M, we have $\mathrm{Con}_{\mathbb{I}}(\mathcal{C}_2 \times M) \cong \mathrm{Con}_{\mathbb{I}}(\mathrm{B}(M)) \cong \mathcal{C}_2 \times \mathrm{Con}_{\mathbb{I}}(M)$, while $\mathrm{Con}(\mathcal{C}_2 \times M) \cong \mathcal{C}_2 \times \mathrm{Con}(M)$ and $\mathrm{Con}(\mathrm{B}(M)) \cong \mathcal{C}_2^2 \times \mathrm{Con}(M)$.

Recall that the pseudo–Kleene algebras $L_{\nu,\kappa}$ from Remark 10 have $\mathrm{Con}(L_{\nu,\kappa}) \cong \mathrm{Con}_{\mathbb{I}}(L_{\nu,\kappa})^2$, so that $|\mathrm{Con}(L_{\nu,\kappa})| = \kappa^2$. Now let us obtain such pseudo–Kleene algebras with κ many (involution–preserving) congruences, and, moreover, with their congruences coinciding to those of their lattice reducts:

Theorem 2 *For any infinite cardinal number ν, any cardinal number κ with $2 \leq \kappa \leq \nu$ or $\kappa = 2^\nu$ and each $\mu \in \{\nu, 2^\nu\}$:*

- *there exists a bounded lattice $L_{\nu,\mu,\kappa}$ with $|\mathrm{Filt}(L_{\nu,\mu,\kappa})| = |\mathrm{Id}(L_{\nu,\mu,\kappa})| = \mu$ and $|\mathrm{Con}(L_{\nu,\mu,\kappa})| = \kappa$;*

- *if $\kappa \leq \nu$, then, for any well–ordered set $(S, \leq)$ with largest element 1^S having $|S| = \kappa$, there exists a lattice $L_{\nu,\mu,\kappa}$ with $|\mathrm{Filt}(L_{\nu,\mu,\kappa})| = |\mathrm{Id}(L_{\nu,\mu,\kappa})| = \mu$ and such that the lattice $\mathrm{Con}(L_{\nu,\mu,\kappa})$ isomorphic to $(S, \leq)$; moreover, if the largest element 1^S of $(S, \leq)$ has a predecessor (that is, if 1^S is strictly join–irreducible in the bounded chain $(S, \leq)$), then $L_{\nu,\mu,\kappa}$ can be chosen to be a bounded lattice;*

- *there exists a bounded involution lattice $L_{\nu,\mu,\kappa}$ with $|\mathrm{Filt}(L_{\nu,\mu,\kappa})| = |\mathrm{Id}(L_{\nu,\mu,\kappa})| = \mu$ and $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,\mu,\kappa})| = |\mathrm{Con}(L_{\nu,\mu,\kappa})| = \kappa$; moreover, $L_{\nu,\mu,\kappa}$ can be chosen to be a pseudo–Kleene algebra and, if $\kappa = 2^\nu$, even a Kleene algebra, more precisely a Kleene chain;*

- *if $\kappa \leq \nu$, then, for any well–ordered set $(S, \leq)$ with largest element 1^S having $|S| = \kappa$, there exists an involution lattice $L_{\nu,\mu,\kappa}$ with $|\mathrm{Filt}(L_{\nu,\mu,\kappa})| = |\mathrm{Id}(L_{\nu,\mu,\kappa})| = \mu$ and such that $\mathrm{Con}_{\mathbb{I}}(L_{\nu,\mu,\kappa}) = \mathrm{Con}(L_{\nu,\mu,\kappa}) \cong (S, \leq)$ and, furthermore, $L_{\nu,\mu,\kappa}$ can be chosen such that it satisfies condition* Ⓚ*; moreover, if the largest element 1^S of*

$(S,\leq)$ has a predecessor (that is, if 1^S is strictly join–irreducible in the bounded chain $(S,\leq)$), then $L_{\nu,\mu,\kappa}$ can be chosen to be a bounded involution lattice, thus even a pseudo–Kleene algebra.

Proof: Let T be a set with $|T|=\nu$ and let us consider the orthomodular lattice and thus pseudo–Kleene algebra $L_{\nu,\nu,2}=\mathcal{M}_\nu=\mathcal{M}_{\nu+\nu}=\boxplus_{t\in T}\mathcal{C}_2^2$, which has length 3 and a simple lattice reduct, thus all filters and ideals principal and $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,\nu,2})|=|\mathrm{Con}(L_{\nu,\nu,2})|=2$. By Corollary 1, for every cardinal number $3\leq\kappa\leq\nu$ and every ordinal ι having $|\iota|=\kappa$, there exists an involution lattice $L_{\nu,\nu,\kappa}$ that satisfies ⓚ and has $|\mathrm{Filt}(L_{\nu,\nu,\kappa})|=|\mathrm{Id}(L_{\nu,\nu,\kappa})|=|\mathrm{Filt}(L_{\nu,\nu,2})|=|\mathrm{Id}(L_{\nu,\nu,2})|=\nu$ and $\mathrm{Con}_{\mathbb{I}}(L_{\nu,\nu,\kappa})=\mathrm{Con}(L_{\nu,\nu,\kappa})\cong\{\lambda:1\leq\lambda\leq\iota\}$, so that $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,\nu,\kappa})|=|\mathrm{Con}(L_{\nu,\nu,\kappa})|=\kappa$. Additionally, if ι is a successor ordinal, then $L_{\nu,\nu,\kappa}$ is bounded, thus it is a pseudo–Kleene algebra.

Now let C be a well–ordered set with top element having $|C|=\nu$, as in Example 1, so that the Kleene chain $L_{\nu,\nu,2^\nu}=C\oplus C^d$ has $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,\nu,2^\nu})|=|\mathrm{Con}(C)|=2^\nu$ and $|\mathrm{Filt}(L_{\nu,\nu,2^\nu})|=|\mathrm{Id}(L_{\nu,\nu,2^\nu})|=\nu$.

Now we consider the pseudo–Kleene algebras $L_{\nu,2^\nu,2^\nu}=M_{\nu,2^\nu}\oplus M_{\nu,2^\nu}^d$ and $L_{\nu,2^\nu,2}=(M_{\nu,2^\nu}\oplus M_{\nu,2^\nu}^d)\boxplus\mathcal{C}_2^2$, where $M_{\nu,2^\nu}$ is the bounded lattice M of cardinality ν from Example 2. Since $|M_{\nu,2^\nu}|=\nu$, we have $|L_{\nu,2^\nu,2^\nu}|=|L_{\nu,2^\nu,2}|=\nu$. Since $|\mathrm{Filt}(M_{\nu,2^\nu})|=\nu$ and $|\mathrm{Id}(M_{\nu,2^\nu})|=2^\nu$, we have $|\mathrm{Filt}(L_{\nu,2^\nu,2^\nu})|=|\mathrm{Id}(L_{\nu,2^\nu,2^\nu})|=|\mathrm{Filt}(L_{\nu,2^\nu,2})|=|\mathrm{Id}(L_{\nu,2^\nu,2})|=2^\nu$. Finally, $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,2^\nu,2^\nu})|=|\mathrm{Con}(M_{\nu,2^\nu})|=2^\nu=2^\nu\cdot 2^\nu=|\mathrm{Con}(L_{\nu,2^\nu,2^\nu})|$ and, since $M_{\nu,2^\nu}$ is 0–regular, so that $\mathrm{Con}_{01}(M_{\nu,2^\nu}\oplus M_{\nu,2^\nu}^d)=\{\Delta_{M_{\nu,2^\nu}\oplus M_{\nu,2^\nu}^d}\}$, we have $\mathrm{Con}_{\mathbb{I}}(L_{\nu,2^\nu,2})=\mathrm{Con}(L_{\nu,2^\nu,2})=\{\Delta_{L_{\nu,2^\nu,2}},\nabla_{L_{\nu,2^\nu,2}}\}$.

By Corollary 1, it follows that, for every cardinal number $2\leq\kappa\leq\nu$ and any ordinal ι with $|\iota|=\kappa$, there exists an involution lattice $L_{\nu,2^\nu,\kappa}$ that satisfies ⓚ and has $|\mathrm{Filt}(L_{\nu,2^\nu,\kappa})|=|\mathrm{Id}(L_{\nu,2^\nu,\kappa})|=|\mathrm{Filt}(L_{\nu,2^\nu,2})|=|\mathrm{Id}(L_{\nu,2^\nu,2})|=2^\nu$ and $\mathrm{Con}_{\mathbb{I}}(L_{\nu,2^\nu,\kappa})=\mathrm{Con}(L_{\nu,2^\nu,\kappa})\cong\{\lambda:1\leq\lambda\leq\iota\}$, so that $|\mathrm{Con}_{\mathbb{I}}(L_{\nu,2^\nu,\kappa})|=|\mathrm{Con}(L_{\nu,2^\nu,\kappa})|=\kappa$. Additionally, if ι is a successor ordinal, then $L_{\nu,2^\nu,\kappa}$ is bounded, thus it is a pseudo–Kleene algebra. □

Corollary 2 *For any infinite cardinal number ν, any cardinal number κ with $2\leq\kappa\leq\nu$ or $\kappa=2^\nu$ and each $\mu\in\{\nu,2^\nu\}$, there exists an antiortholattice $A_{\nu,\mu,\kappa}$ with $|\mathrm{Filt}(A_{\nu,\mu,\kappa})|=|\mathrm{Id}(A_{\nu,\mu,\kappa})|=\mu$ and $|\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,\mu,\kappa})|=\kappa$. Furthermore, if $3\leq\kappa\leq\nu$, then, for every successor ordinal τ with $|\tau|=\kappa$ and whose predecessor is also a successor ordinal, we may choose $A_{\nu,\mu,\kappa}$ such that $\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,\mu,\kappa})=\mathrm{Con}_{01}(A_{\nu,\mu,\kappa})\cup\{\nabla_{A_{\nu,\mu,\kappa}}\}\cong\{\lambda:1\leq\lambda\leq\tau\}$; otherwise written: if $3\leq\kappa\leq\nu$, then, for any well–ordered set $(S,\leq)$ of* *cardinality* κ *and having a largest element that has a predecessor* p *such that* p *has a predecessor in* $(S, \leq)$*, as well, we may choose* $A_{\nu,\mu,\kappa}$ *such that* $\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,\mu,\kappa}) = \mathrm{Con}_{01}(A_{\nu,\mu,\kappa}) \cup \{\nabla_{A_{\nu,\mu,\kappa}}\} \cong (S, \leq)$.

Proof: Let us consider the 0–regular bounded lattice with ν elements and as many ideals as subsets $M_{\nu,2^\nu}$ from the proof of Theorem 2, which equals M from Example 2. Then the antiortholattice $A_{\nu,2^\nu,2} = M_{\nu,2^\nu} \oplus M^d_{\nu,2^\nu}$ has $|A_{\nu,2^\nu,2}| = \nu$, $|\mathrm{Filt}(A_{\nu,2^\nu,2})| = |\mathrm{Id}(A_{\nu,2^\nu,2})| = 2^\nu$ and $|\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,2^\nu,2})| = 2$.

Now we consider the modular lattice $\mathcal{M}_\nu$ with ν elements and length 3, which is simple, thus 0–regular, and has all filters and ideals principal since it has finite length, and we let $A_{\nu,\nu,2} = \mathcal{M}_\nu \oplus \mathcal{M}^d_\nu$. Then the antiortholattice $A_{\nu,\nu,2}$ has $|A_{\nu,\nu,2}| = \nu$, $|\mathrm{Filt}(A_{\nu,\nu,2})| = |\mathrm{Id}(A_{\nu,\nu,2})| = \nu$ and $|\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,\nu,2})| = 2$.

Now let κ be a cardinal number with $3 \leq \kappa \leq \nu$ or $\kappa = 2^\nu$, let $\mu \in \{\nu, 2^\nu\}$, and consider the antiortholattice $A_{\nu,\mu,\kappa} = \mathcal{C}_2 \oplus L_{\nu,\mu,\kappa-1} \oplus \mathcal{C}_2$, where $L_{\nu,\mu,\kappa-1}$ is a pseudo–Kleene algebra as in Theorem 2, that is with ν elements, μ filters and ideals and $\kappa - 1$ congruences. Then $|A_{\nu,\mu,\kappa}| = \nu$, $|\mathrm{Filt}(A_{\nu,\mu,\kappa})| = |\mathrm{Id}(A_{\nu,\mu,\kappa})| = \mu$ and $|\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,\mu,\kappa})| = |\mathrm{Con}_{\mathbb{BI}01}(A_{\nu,\mu,\kappa})| + 1 = |\mathrm{Con}_{\mathbb{I}}(L_{\nu,\mu,\kappa-1})| + 1 = \kappa - 1 + 1 = \kappa$.

Additionally, if $\kappa \leq \nu$ and thus $2 \leq \kappa - 1 \leq \nu$, then, according to Theorem 2, for every successor ordinal σ with $|\sigma| = \kappa - 1$, $L_{\nu,\mu,\kappa-1}$ can be chosen such that all its lattice congruences preserve its involution and its congruence lattice is isomorphic to $\{\lambda : 1 \leq \lambda \leq \sigma\}$. Now, if we let $\tau = \sigma + 1$, so that $|\tau| = \kappa$, then it follows that $\mathrm{Con}_{\mathbb{BZL}}(A_{\nu,\mu,\kappa}) = \mathrm{Con}_{\mathbb{BI}0}(A_{\nu,\mu,\kappa}) \cup \{\nabla_{A_{\nu,\mu,\kappa}}\} = \{eq(L_{\nu,\mu,\kappa-1}/\beta \cup \{\{0\}, \{1\}\}) : \beta \in \mathrm{Con}_{\mathbb{I}}(L_{\nu,\mu,\kappa-1})\} \cup \{\nabla_{A_{\nu,\mu,\kappa}}\} = \{eq(L_{\nu,\mu,\kappa-1}/\beta \cup \{\{0\}, \{1\}\}) : \beta \in \mathrm{Con}(L_{\nu,\mu,\kappa-1})\} \cup \{\nabla_{A_{\nu,\mu,\kappa}}\} = \mathrm{Con}_{01}(A_{\nu,\mu,\kappa}) \cup \{\nabla_{A_{\nu,\mu,\kappa}}\} \cong \{\lambda : 1 \leq \lambda \leq \sigma\} \oplus \mathcal{C}_2 \cong \{\lambda : 1 \leq \lambda \leq \sigma + 1\} = \{\lambda : 1 \leq \lambda \leq \tau\}$. □

Corollary 3 *Under the Generalized Continuum Hypothesis:*

- *an infinite (bounded) lattice with any numbers of filters and ideals can have any number of congruences between* 2 *and its number of subsets;*
- *a (bounded) involution lattice and even a pseudo–Kleene algebra with any number of ideals can have any number of congruences between* 2 *and its number of subsets and, simultaneously, when it has strictly less congruences than subsets, its congruences coinciding to those of its lattice reduct;*

- *an antiortholattice with any number of ideals can have any number of congruences between* 2 *and its number of subsets and, simultaneously, when it has strictly less congruences than subsets, its proper congruences coinciding to the congruences of its lattice reduct that have singleton classes of its lattice bounds.*

Under the Continuum Hypothesis, the above hold for countable (bounded) lattices, countable (bounded) involution lattices and even pseudo–Kleene algebras, respectively countable antiortholattices.

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