

A Complete Axiomatisation for Probabilistic Trace Equivalence

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Abstract

We provide an axiomatisation for $=_{pTr}$, a variant of probabilistic trace equivalence as formulated by Bernardo et al., 2014, in the setting of the alternating model of Hansson. The equivalence considers traces individually instead of trace distributions. We show that our axiomatisation is sound and also complete for recursion-free sequential processes. Due to the nature of the trace equivalence, the axiomatisation is particularly complex.

Keywords: Probabilistic processes, Trace equivalence, Complete axiomatisation.

1 Introduction

Non-deterministic probabilistic labelled transition systems (which we call *probabilistic systems* for short) combine choice, as is modelled with non-deterministic finite state machines, and probabilities, as is modelled with Markov chains, in a single model. This is useful to model concurrent systems where interaction leads to probabilistic behaviour. The behaviour of such systems can be explained in terms of a set of observations, where an observation is a probabilistic trace, a sequence of actions allowed by

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the system with a certain probability. To determine if two systems are equal, their sets of probabilistic traces must be equal. We are interested in the equivalence relation this notion yields. In particular, we are inspired by the equivalence relation \sim_{PT} formulated by Bernardo et al.[4], and more specifically, their alternative characterisation based on weighted traces (Definition 3.2 and Theorem 3.8, respectively, of that article). We apply this notion to the alternating model of probabilistic systems [13], and ask ourselves how it can be axiomatised.

In systems where non-determinism and probabilities are combined, the semantics gain an extra dimension. Let us illustrate this by a thought experiment.

Example 1.1. Imagine we have a game with two identical-looking coins. One coin is fair, the other always produces “heads”. The player chooses one of the coins and then tosses it. We can describe this game as a probabilistic system (Figure 1).

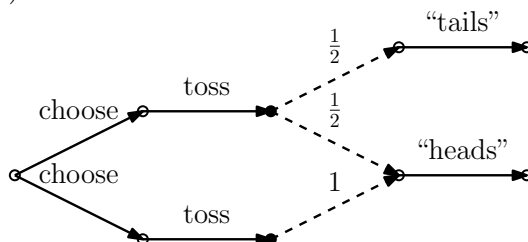


Figure 1: A coin game

The system in Figure 1 begins with a non-deterministic choice, but we could also have modelled it as a probabilistic chance. If the coins are truly identical, do we not have an equal chance of picking one over the other? What if the fair coin is always to the left? What if we always pick the coin that already shows “heads”? The fact we choose to model the game using non-determinism means that we have abstracted away from any assumptions and consider all outcomes as coexistent.

A probabilistic trace weighs the chance that a certain sequence of actions is allowed by a system. A trace accumulates the probabilities part of all paths that are the result of the same choices made, of a particular outcome. We assume that all transitions are independent events. So, by probability theory, the weight is the product of these probabilities. Due to the non-determinism, a trace can, in our setting, be associated with more than one weight.

Looking back at Example 1.1, some of the traces are [choose] with weight 1, [choose, toss, “tails”] with weight 1/2, and [choose, toss, “heads”] with weights 1 and 1/2. It is clear that, in our regard, this game is not trace equivalent to a game where both coins are fair. But a game where we choose (non-deterministically) between arbitrarily many fair coins would be trace equivalent to a game where we choose between just two fair coins.

Process languages allow us to specify systems syntactically. We describe probabilistic processes in one of the simplest conceivable probabilistic process languages. The process that does not perform any action is denoted by δ , following the ACP tradition [2]. The prefix operator $a.x$ describes the system that initially allows an a action, after which it continues as the system described by x . Alternative composition, denoted by a plus-sign, specifies that there is a choice between two sub-expressions. For example, the process $b.x + c.y$ describes the system where either a b or c action may occur initially. Finally, probabilistic composition, denoted by circle-plus, specifies that there is a chance distributed among two or more sub-expressions. For instance, the process $\frac{1}{2}b.x \oplus \frac{1}{2}c.y$ describes a system where there is a fifty-fifty chance either a b or c action is allowed initially.

Using operational semantics, we define how a probabilistic system is derived from a probabilistic process. This allows us to extend probabilistic trace equivalence to probabilistic processes. It is in general quite laborious to show that two processes are trace equivalent. It is sometimes far more convenient to have a set of axioms available that allow to transform one probabilistic process expression into another in a stepwise manner. Another important aspect of such axioms is that they provide an alternative view on an equivalence helping to understand whether the equivalence is sufficiently elegant and the right notion for the job at hand.

We present an axiomatisation for probabilistic processes regarding probabilistic trace equivalence, which we prove is sound in general, and complete for recursion-free sequential processes. We do this by a normal form for probabilistic processes. Any process can be rewritten to this normal form. Furthermore, we show that two processes in normal form are syntactically equal iff they are probabilistic trace equivalent.

The presented axioms are particularly complex. We give several counter-examples that show that simpler axioms are not sound, which shows that our proposed axioms are likely to be the natural axioms belonging to the trace equivalence as proposed in Bernardo et al. [4]. A natural question arising from this, which we will not answer in this paper, is whether the proposed trace equivalence is the right one.

1.1 Related Work

Earlier, we gave an example of how we might model using probabilistic systems, however, there are many different approaches when it comes to specifying one. In *generative systems* [11], probabilities are distributed among all transitions originating from the same state, while in *reactive systems* [16], only among identically labelled transitions. Note that in both variants, the probabilities replace non-determinism rather than add to it. In the *alternating model* of Hansson [13, 14], transitions alternate between non-deterministic and probabilistic states. Non-deterministic states have transitions with action labels. Probabilities are distributed among transitions originating from probabilistic states. In Segala systems [22, 21], transitions are probability distributions ranging over actions paired with target states, and similarly in *simple Segala systems*, ranging over the same action paired with target states. Non-determinism is preserved by the fact that multiple transitions may originate from the same state. An example of both models can be seen in Figure 2. We note that some approaches are compatible to others. These systems can be translated to the compatible form under bisimulation [23]. Sokolova and de Vink have presented a hierarchy of compatible systems [25]. The alternating model of Hansson and simple

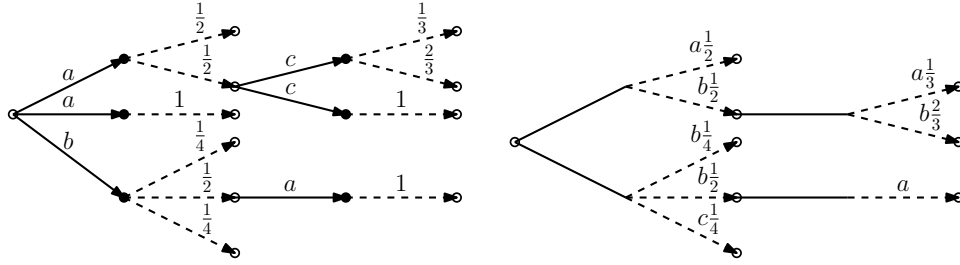


Figure 2: An alternating system (left) and a Segala system (right)

Segala systems may appear similar, however, they are not compatible under weak bisimulation, as Bandini and Segala pointed out [1].

We focus on trace semantics. There are many different approaches when it comes to defining traces in a probabilistic setting. Many approaches introduce the notion of an adversary (or scheduler). This adversary resolves non-deterministic behaviour by either assigning probabilities to the different choices or by determinising them. The resolutions can be aggregated in terms of a distribution of traces. Two systems are trace equivalent if for each resolution in the one, a resolution in the other exists, where the trace distri-

butions are equal. Segala has defined this using simple Segala systems [20]. Parma and Segala gave a sound and complete axiomatisation [19] for the coarsest congruence contained in this notion, which rather is a simulation equivalence [17]. More recently, notions like supremal probabilities [5] and coherent resolutions [3] were proposed.

Bernardo et al. noted that equivalences based on probabilistic trace distributions have undesirable properties [4]. It can differentiate models that allow the same sequence of actions, with the same probability. Also, it is not a congruence for the parallel composition. Instead, they opted for a coarser kind of trace semantics for simple Segala systems, where each trace is considered individually instead of entire trace distributions. To our knowledge, no axiomatisation exists for this equivalence as it stands.

For notational ease, rather than simple Segala systems, we study the axiomatisation for the trace semantics proposed by Bernardo et al. in the alternating model of Hansson. We hope this makes the axioms, and their soundness and completeness proofs, less complicated. We also hope that our axiomatisation provides inspiration to find one in the setting of simple Segala systems.

2 Probabilistic Processes

We define a probabilistic process algebra which is an extension to CCS [18], inspired by the work of Bandini and Segala [1]. It includes deadlocks, sequencing and choice. The sequential composition is guarded by an action (or label), but it does not terminate into a process. Rather it ends in a discrete distribution of processes, each process preceded by a probability. In this section, we give the definition of the language and a few helpful notations used in the remainder of this article.

We assume the existence of a finite set of atomic actions \mathcal{A} , and we refer to probabilities as positive fractions up to including 1, that is numbers from the domain $\mathcal{P} = \mathbb{Q} \cap (0, 1]$.

Definition 2.1. We define a *probabilistic process expression* by the following syntax

$$\begin{aligned} N &::= \delta \mid \alpha.P(1) \mid N + N \\ P(\rho) &::= \rho N \mid P(\rho r) \oplus P(\rho - \rho r) \end{aligned}$$

where $\alpha \in \mathcal{A}$ is an action, $\rho \in \mathcal{P}$ is a probability, and $r \in \mathbb{Q} \cap (0, 1)$ is a ratio. Furthermore, we have the following operators: deadlock (δ), sequential

composition ($\alpha.P(1)$), alternative composition ($N_1 + N_2$), and probabilistic composition ($\rho_1 N_1 \oplus \cdots \oplus \rho_n N_n$). In the sequel, we also use N as the set of all expressions accepted by the syntax of N . Similarly, we also use $P(\rho)$. We write \mathbf{P} as the collective of all possible P -expressions, that is $\mathbf{P} = \bigcup_{\rho \in \mathcal{P}} P(\rho)$.

For convenience, we would like to quantify over the alternative and probabilistic composition. Hence, we introduce respectively the sum and circle-sum operators.

Definition 2.2. Let $X \in N^*$ be $[x_1, \dots, x_n]$, a *finite* sequence of processes. Then we write

$$\sum_{y \in X} y = x_1 + \cdots + x_n$$

or in case X is empty

$$\sum_{y \in X} y = \delta.$$

Definition 2.3. Let $[\rho_1, \dots, \rho_n]$ be a sequence of probabilities and let $[x_1, \dots, x_n]$ be a sequence of processes, both *finite* and *non-empty*, where $I = \{1, \dots, n\}$ is their index set. Then we write

$$\sum_{i \in I} \rho_i x_i = \rho_1 x_1 \oplus \cdots \oplus \rho_n x_n.$$

Remark 2.4. Note the difference in indexing between \sum and \sum . The sequence-based approach, used for \sum , aides some of the proofs in Section 6.

From the syntax, it becomes clear that any P -expression consists of one or more ρx terms. Later we show that the \oplus -operator is associative and commutative. In that case, any P -expression can be seen as a sequence of (ρ, x) pairs, which can be expressed using the circle-sum operator.

Lemma 2.5. Any probabilistic process expression $u \in \mathbf{P}$ can be expressed as a probabilistic sum, that is

$$u = \sum_{i \in I} \rho_i x_i$$

for some sequence of probabilities ρ and some sequence of processes x .

When reasoning about probabilistic processes parts of a distribution may often end up in a deadlock. Therefore, we introduce a special notation that abbreviates this, called a *junction*.

Definition 2.6. Let $u_\rho \in P(\rho)$ be a probabilistic process expression. Then we write

$$\downarrow u_\rho = \begin{cases} u_\rho \oplus (1 - \rho) \delta & \text{if } \rho < 1, \text{ and} \\ u_\rho & \text{if } \rho = 1. \end{cases}$$

For instance $\alpha.(\frac{1}{3}x \oplus \frac{2}{3}\delta)$ is written as $\alpha.\downarrow\frac{1}{3}x$.

3 Alternating Probabilistic Labelled Transition Systems

With each probabilistic process, we associate an alternating probabilistic labelled transition system (APLTS), which extends LTSs with probabilistic transitions. The term *alternating* refers to the fact that APLTSs strictly alternate between non-deterministic states and probabilistic states. Non-deterministic states may have no outgoing transitions, while probabilistic states have at least one outgoing transition. The initial state is a non-deterministic state.

Definition 3.1. An *alternating probabilistic labelled transition system* (APLTS) is a 6-tuple $(S_N, S_P, \mathcal{A}, \rightarrow, \dashrightarrow, s_0)$ where

- S_N is a set of *non-deterministic states*,
- S_P is a set of *probabilistic states*,
- \mathcal{A} is a set of *action labels*,
- $\rightarrow \subseteq S_N \times \mathcal{A} \times S_P$ is a *non-deterministic transition relation*,
- $\dashrightarrow : (S_P \times \mathcal{P} \times S_N) \rightarrow \mathbb{N}$ is a *probabilistic transition relation*, which is a multiset
 - where $\mathcal{P} = \mathbb{Q} \cap (0, 1]$ is a set of probabilities, and
 - for all states $s \in S_P$ it holds that $\sum_{s \dashrightarrow v}^{\rho:n} \rho \cdot n = 1$, thus forming a discrete distribution,
- and $s_0 \in S_N$ is the *initial state*.

Remark 3.2. We use a shorthand notation and write $s \xrightarrow{\alpha} t$ instead of $(s, \alpha, t) \in \rightarrow$, and similarly use $s \dashrightarrow t$ instead of $(s, \rho, t, n) \in \dashrightarrow$.

$$\begin{array}{c}
\frac{N_1 \xrightarrow{\alpha} \mathbf{P}}{N_1 + N_2 \xrightarrow{\alpha} \mathbf{P}} \quad \frac{N_2 \xrightarrow{\alpha} \mathbf{P}}{N_1 + N_2 \xrightarrow{\alpha} \mathbf{P}} \quad \frac{}{\alpha.P(1) \xrightarrow{\alpha} P(1)} \\
\frac{\mathbf{P}_1 \xrightarrow{\rho:n} N \quad \mathbf{P}_2 \xrightarrow{\rho:m} N}{\mathbf{P}_1 \oplus \mathbf{P}_2 \xrightarrow{\rho:n+m} N} \quad \frac{}{\rho N \xrightarrow{\rho:1} N} \quad \frac{\forall n \in \mathbb{N}_{>0} \mathbf{P} \xrightarrow{\rho:n} N}{\mathbf{P} \xrightarrow{\rho:0} N}
\end{array}$$

Table 1: Operational semantics of probabilistic process expressions

Definition 3.3. Let $x \in N$ be a probabilistic process containing actions from \mathcal{A} . We define $\llbracket x \rrbracket$, the semantics of x , as an APLTS $(S_N, S_P, \mathcal{A}, \rightarrow, \dashrightarrow, s_0)$ where

- the set of states S_N contains all N -expressions,
- the set of states S_P contains all \mathbf{P} -expressions,
- the \rightarrow and \dashrightarrow transitions are formed inductively using the operational semantics as defined in Table 1, which are stratifiable and therefore consistent [12],
- and the initial state s_0 is x .

Example 3.4. Assume that $\mathcal{A} = \{a, b, c, d\}$ and let $x \in N$ be a process defined as

$$a. \left(\frac{1}{2}c.1\delta \oplus \frac{1}{3}c.1\delta \oplus \frac{1}{6}d.1\delta \right) + a. \downarrow \frac{1}{2}c.1\delta + b.1c.1\delta + \delta.$$

Using Definition 3.3 we find the corresponding APLTS as shown in Figure 3.

4 Probabilistic Trace Equivalence

In this section we introduce the notion of p -traces, the set of probabilistic traces for an APLTS, as well as a number of useful operations on and properties of p -traces. We also define the notion of *probabilistic trace equivalence* for processes.

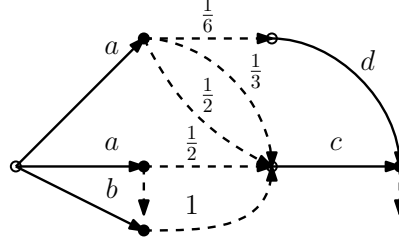


Figure 3: The APLTS $\llbracket a. (\frac{1}{2}c.1\delta \oplus \frac{1}{3}c.1\delta \oplus \frac{1}{6}d.1\delta) + a.\downarrow\frac{1}{2}c.1\delta + b.1c.1\delta + \delta \rrbracket$
Note that, for brevity, the residual probabilities to the deadlock state are abbreviated as a downward arrow.

Definition 4.1. A *weighted trace* or *p-trace* is a pair consisting of a sequence of actions from \mathcal{A}^* and a probability from $\mathcal{P} = (0, 1] \cap \mathbb{Q}$. The set of all *p-trace* sets is $\mathcal{T} = 2^{\mathcal{A}^* \times \mathcal{P}}$.

Definition 4.2. Let $X, Y \in \mathcal{T}$ be *p-trace* sets, $\alpha \in \mathcal{A}$ be an action, $\rho \in \mathcal{P}$ be a probability, and let ε be the empty sequence. We define the following operators on sets of *p-traces*

$$\begin{aligned} \alpha.X &= \{(\alpha\sigma, \rho) \mid (\sigma, \rho) \in X\} & \rho \cdot X &= \{(\sigma, \rho \cdot \varrho) \mid (\sigma, \varrho) \in X\} \\ &\cup \{(\varepsilon, \rho) \mid (\varepsilon, \rho) \in X\} & X_\varepsilon &= \{(\varepsilon, \rho) \mid (\varepsilon, \rho) \in X\} \end{aligned}$$

$$\begin{aligned} X \uplus Y &= \{(\sigma, \rho + \varrho) \mid (\sigma, \rho) \in X \wedge (\sigma, \varrho) \in Y\} & X_{\bar{\varepsilon}} &= X \setminus X_\varepsilon \\ &\cup \{(\sigma, \rho) \mid (\sigma, \rho) \in X \wedge \forall \varrho (\sigma, \varrho) \notin Y\} \\ &\cup \{(\sigma, \varrho) \mid (\sigma, \varrho) \in Y \wedge \forall \rho (\sigma, \rho) \notin X\} \end{aligned}$$

Let $T \in \mathcal{T}^*$ be a sequence of *p-trace* sets. We define $\biguplus_{i \in I} T_i = T_1 \uplus \dots \uplus T_n$.

Example 4.3. As a demonstration of the operators just defined, we witness that

$$\begin{aligned} &\rho \cdot \alpha. \{(\varepsilon, 1)\} \uplus (1 - \rho) \cdot \{(\varepsilon, 1)\} \\ &= \rho \cdot \{(\varepsilon, 1), (\alpha, 1)\} \uplus (1 - \rho) \cdot \{(\varepsilon, 1)\} \\ &= \{(\varepsilon, \rho), (\alpha, \rho)\} \uplus \{(\varepsilon, 1 - \rho)\} \\ &= \{(\varepsilon, 1), (\alpha, \rho)\} \end{aligned}$$

and

$$\begin{aligned}
& (\alpha \cdot \{(\varepsilon, 1)\})_{\varepsilon} \cup (\rho \cdot \alpha \cdot \{(\varepsilon, 1)\})_{\bar{\varepsilon}} \\
&= \{(\varepsilon, 1), (\alpha, 1)\}_{\varepsilon} \cup \{(\varepsilon, \rho), (\alpha, \rho)\}_{\bar{\varepsilon}} \\
&= \{(\varepsilon, 1)\} \cup \{(\alpha, \rho)\} \\
&= \{(\varepsilon, 1), (\alpha, \rho)\}
\end{aligned}$$

Definition 4.4. Let $(S_N, S_P, \mathcal{A}, \rightarrow, \dashrightarrow, s_0)$ be an APLTS. We define the function $pTr : (S_N \cup S_P) \rightarrow \mathcal{T}$ such that for all states $s \in S_N \cup S_P$, $pTr(s)$ is the minimal set such that

- if $s \in S_N$ and s has no outgoing transitions, then $(\varepsilon, 1) \in pTr(s)$, and
- if $s \in S_N$ and s has at least one outgoing transition, then $\bigcup_{(\alpha, t) | s \dashrightarrow t} \alpha \cdot pTr(t) \subseteq pTr(s)$, and
- if $s \in S_P$ then $\biguplus_{(\rho, t, n) | s \dashrightarrow^{\rho, n} t} \biguplus_{i=1}^n \rho \cdot pTr(t) \subseteq pTr(s)$.

This function forms the basis of our equivalence. Two processes are said to be p -trace equivalent iff their derived APLTSs are characterised by the same set of p -traces.

Definition 4.5. Let $s, t \in S_N \cup S_P$ be two states. We call s and t p -trace equivalent iff $pTr(s) = pTr(t)$, which we denote as $s =_{pTr} t$. We call two APLTSs p -trace equivalent iff their initial states are so.

Lemma 4.6. p -trace equivalence is an equivalence relation.

We extend the equivalence to probabilistic processes, and write $\llbracket x \rrbracket =_{pTr} \llbracket y \rrbracket$ as $x =_{pTr} y$. Likewise, we refer to $pTr \llbracket x \rrbracket$ simply as the set of p -traces of a process x .

Example 4.7. Let $x \in N$ be the process

$$a. \left[\frac{1}{2} \left(b. \downarrow_{\frac{1}{8}} c. 1\delta + b. \downarrow_{\frac{3}{8}} c. 1\delta \right) \oplus \frac{1}{2} \left(b. \downarrow_{\frac{1}{8}} c. 1\delta + b. \downarrow_{\frac{5}{8}} c. 1\delta \right) \right].$$

In Figure 4 we see the corresponding APLTS $\llbracket x \rrbracket$. The set of p -traces

$$\begin{aligned}
& \text{of process } x \text{ is } pTr \llbracket x \rrbracket = \left\{ \begin{array}{l} (\varepsilon, 1), (a, 1), \left(ab, \frac{1}{2} + \frac{1}{2} \right), \\ \left(abc, \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{1}{8} \right), \left(abc, \frac{1}{2} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{1}{8} \right), \\ \left(abc, \frac{1}{2} \cdot \frac{1}{8} + \frac{1}{2} \cdot \frac{5}{8} \right), \left(abc, \frac{1}{2} \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{5}{8} \right) \end{array} \right\} \\
&= \{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{8}), (abc, \frac{2}{8}), (abc, \frac{3}{8}), (abc, \frac{4}{8})\}.
\end{aligned}$$

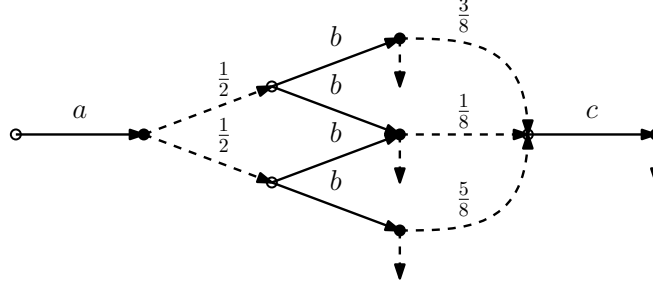


Figure 4: The APLTS $\left[\left[a. \left[\frac{1}{2} \left(b. \downarrow \frac{1}{8} c. 1\delta + b. \downarrow \frac{3}{8} c. 1\delta \right) \oplus \frac{1}{2} \left(b. \downarrow \frac{1}{8} c. 1\delta + b. \downarrow \frac{5}{8} c. 1\delta \right) \right] \right]$

When we explore p -trace sets derived from probabilistic processes (N), four properties come to our attention: the trace $(\varepsilon, 1)$ is always present, a trace with an action sequence of length one always has the probability 1, the complete set of prefixes (of its action sequence) are present for each trace, and each trace that is a prefix to another trace has a probability greater than or equal to this trace. We can extend some of these properties to p -trace sets generated from process expressions in general (N and P).

Lemma 4.8. All p -trace sets derived from probabilistic process expressions are ε -preserving: let $x \in N$ and $u_\rho \in P(\rho)$ be process expressions. Then $pTr[x]_\varepsilon = \{(\varepsilon, 1)\}$ and $pTr[u_\rho]_\varepsilon = \{(\varepsilon, \rho)\}$.

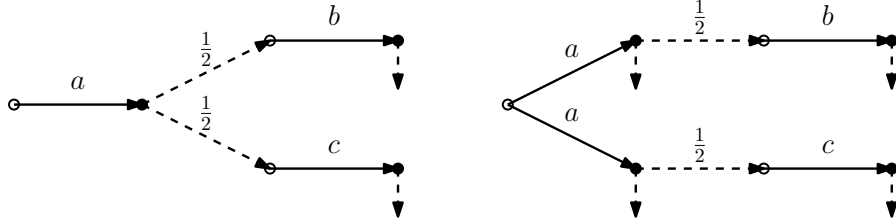
We notice a relation between the ρ -parameter in a process expression and the probability of the empty trace derived from this expression. We are interested in understanding whether we can find more relations between expressions and their derived p -trace sets. Table 2 shows a number of such properties.

Lemma 4.9. The $=_{pTr}$ relation is a congruence for the complete syntax of process expressions.

Proof: This follows from the properties in Table 2. \square

Remark 4.10. Our process algebra does not include a generic renaming operator, i.e., an operator that replaces all occurrences of an action to another in its sub-expression. However, we note that this operator would not be congruent in our setting. For example, it is clear that the p -trace equivalent processes shown in Figure 5 are no longer equivalent if the b action is renamed to c . Yet, a congruence can be achieved if renaming-to is restricted to actions not already occurring in the process.

P1	$pTr[\delta] = \{(\varepsilon, 1)\}$	
P2	$pTr[\alpha.u] = \alpha.pTr[u]$	
P3	$pTr[x + y] = pTr[x] \cup pTr[y]$	
P4	$pTr\left[\downarrow u\right] = pTr[u]_{\bar{\varepsilon}} \cup \{(\varepsilon, 1)\}$	
P5	$pTr[\rho x] = \rho \cdot pTr[x]$	
P6	$pTr[u \oplus v] = pTr[u] \uplus pTr[v]$	
P7	$pTr\left[\sum_{x \in X} x\right] = \bigcup_{x \in X} pTr[x]$	if X is not empty
P8	$pTr\left[\sum_{i \in I} \rho_i x_i\right] = \biguplus_{i \in I} \rho_i \cdot pTr[x_i]$	

Table 2: Relation between process and p -traces operators.Figure 5: The APLTSs $\llbracket a.(\frac{1}{2}b.1\delta \oplus \frac{1}{2}c.1\delta) \rrbracket$ and $\llbracket a.\downarrow\frac{1}{2}b.1\delta + a.\downarrow\frac{1}{2}c.1\delta \rrbracket$

Remark 4.11. The model we use has a particular property. While in simple Segala systems it would not be possible due to the nature of distributions, our operational semantics allow for multiple transitions from the same probabilistic state to the same nondeterministic state as long as the considered transitions are labelled with different probabilities, otherwise the first rule in the second row of Table 1 would come into play by merging those transitions while summing up their multiplicities. However, Definition 3.1 admits models in which a probabilistic state features identically labelled probabilistic transitions reaching the same nondeterministic state, as illustrated by the APLTS at the top of Figure 6. We have to decide what this means in terms of our probabilistic traces. In case we do not regard probabilistic transitions, both systems in Figure 6 would be equivalent. If we allow each probabilistic

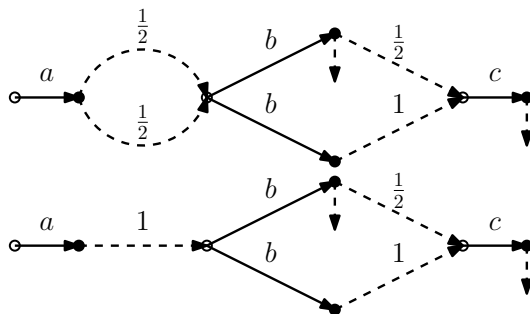


Figure 6: Two alternating probabilistic systems.

branch to be chosen deterministically, not depending on a prior choice, it is possible to do the upper b with probability $1/2$ and the lower b with the same probability in one trace. This would include the trace abc with a weight of $3/4$ in the top system, but not in the bottom system. We have chosen the least conservative (and most discriminative) approach, and chosen the adversary to be deterministic and memoryless. The aforementioned distinction would not arise if, after the considered probabilistic transitions, there would be no internal nondeterministic choice like the one between the two b -transitions.

5 Axioms

In the previous section, we defined an equivalence relation for probabilistic processes based on probabilistic traces. We would like to find a set of axioms, that allows us to manipulate processes in such a way that we find different, but p -trace equivalent, processes. Even more, we would like to be able to find *all* processes that are p -trace equivalent to a given process, using this set of axioms. In other words, we want this set to be sound and complete. We claim we have found such a set. Table 3 shows this axiomatisation. In this section, we give a rationale behind the axioms and in the following subsections, we include a proof for completeness.

5.1 Basic Axioms

For the first few axioms we can draw inspiration from existing axiomatisations of CCS [15], and we can readily adopt the following axioms: alternative composition is associative (A1), commutative (A2), idempotent (A3), and

A1	$x + (y + z) = (x + y) + z$	PA1	$u \oplus v = v \oplus u$
A2	$x + y = y + x$	PA2	$(u \oplus v) \oplus w = u \oplus (v \oplus w)$
A3	$x + x = x$	PA3	$\rho\delta \oplus \varrho\delta = (\rho + \varrho)\delta$
A4	$x + \delta = x$		
PT1 $\alpha.\downarrow\rho(x + y) = \alpha.\downarrow\rho x + \alpha.\downarrow\rho y$			
PT2 $\alpha.\downarrow(\rho r)\beta.\downarrow\varrho x + \alpha.\downarrow\rho\beta.1\delta = \alpha.\downarrow(\rho r)\beta.1\delta + \alpha.\downarrow\rho\beta.\downarrow(\varrho r)x$			
PD1 $\alpha.\sum_{i \in I} \rho_i(x_i + y_i) = \alpha.\sum_{i \in I} \rho_i x_i + \alpha.\sum_{i \in I} \rho_i y_i$			
if x_i and y_j are disjoint [†] , for each i, j			
PD2 $\alpha.\downarrow\sum_{i \in I} \rho_i \left(\sum_{u \in U_i} \beta.u + \sum_{v \in V_i} \beta.v \right) = \alpha.\downarrow\sum_{i \in I} \rho_i \sum_{u \in U_i} \beta.u + \alpha.\downarrow\sum_{i \in I} \rho_i \sum_{v \in V_i} \beta.v$			
if U_i and V_j are disjoint [†] , and not empty, for each i, j			
PD3 $\omega.\downarrow\sum_{i \in I} \rho_i \left(\begin{array}{c} \sum_{u \in U_i} \alpha.\downarrow\varrho_u \beta.u \\ \sum_{v \in V_i} \alpha.\downarrow\varrho_v \beta.v \end{array} + \right) = \left(\begin{array}{c} \omega.\downarrow\sum_{i \in I} \rho_i \sum_{u \in U_i} \alpha.\downarrow\varrho_u \beta.u \\ \omega.\downarrow\sum_{i \in I} \rho_i \sum_{w \in U_i \cup V_i} \alpha.\downarrow\varrho_w \beta.1\delta \\ \omega.\downarrow\sum_{i \in I} \rho_i \sum_{v \in V_i} \alpha.\downarrow\varrho_v \beta.v \end{array} + \right)$			
if U_i and V_j are disjoint [†] , and not empty, for each i, j			
PU1 $\rho\alpha.u \oplus \varrho\alpha.v = (\rho + \varrho)\alpha.\left(\frac{\rho}{\rho + \varrho} \cdot u \oplus \frac{\varrho}{\rho + \varrho} \cdot v\right)$			
PU2 $\rho \sum_{u \in U} \alpha.u \oplus \varrho \sum_{v \in V} \alpha.v = (\rho + \varrho) \sum_{u \in U} \sum_{v \in V} \alpha.\left(\frac{\rho}{\rho + \varrho} \cdot u \oplus \frac{\varrho}{\rho + \varrho} \cdot v\right)$			
if U and V are disjoint [†] , and not empty			
PU3 $\rho \sum_{u \in U} \alpha.\downarrow\rho_u \beta.u \oplus \varrho \sum_{v \in V} \alpha.\downarrow\varrho_v \beta.v = (\rho + \varrho) \sum_{u \in U} \sum_{v \in V} \alpha.\downarrow\left(\frac{\rho\rho_u}{\rho + \varrho} \beta.u \oplus \frac{\varrho\varrho_v}{\rho + \varrho} \beta.v\right)$			
if U and V are disjoint [†] , and not empty			
X1* $\rho \sum_{p \in N} \alpha.\downarrow p\beta.1\delta \oplus \varrho \sum_{q \in P} \alpha.\downarrow q\beta.1\delta = (\rho + \varrho) \sum_{q \in P} \sum_{p \in N} \alpha.\downarrow \frac{\rho p + \varrho q}{\rho + \varrho} \beta.1\delta$			
if N and P are not empty			
X2* $\rho\alpha.\downarrow\varrho\beta.u = (\rho\varrho)\alpha.1\beta.u \oplus (\rho(1 - \varrho))\alpha.1\delta$			

Remark. We use a shorthand notation $\rho \cdot \sum_{i \in I} \rho_i x_i$ instead of $\sum_{i \in I} (\rho \cdot \rho_i) x_i$.

Remark (†). Following the notion of (sets of) disjoint processes given in Definition 5.4.

Remark (*). This is an auxiliary axiom and can be disregarded for the sake of completeness.

Table 3: Axioms of probabilistic trace equivalence

has δ as the identity element (A4). For probabilistic composition we can intuitively find it is commutative (PA1) and associative (PA2). Unlike alternative composition, it is *not* idempotent *nor* is there an identity element, however, we can unify δ terms (PA3). This can be explained by using Remark 4.11 and the fact that something more can be unified in the absence of internal nondeterminism thanks to axioms PU.

5.2 Axioms for p -trace Equivalence

We can also find axioms that hold specifically for p -trace equivalence. For CCS trace equivalence we have the classic law $a.(x + y) = a.x + a.y$ [10]. We can generalise this property for a junction in general obtaining PT1.

If a process only has junctions, rather than more complex probabilistic compositions, we notice an interesting fact. The probabilistic part of the generated traces is always a multiplication of probabilities in the process. We wonder if we can redistribute this product along the probabilities of a process, and find a p -trace equivalent process. Specifically, we wonder if $\downarrow(\rho r) \alpha. \downarrow \varrho x =_{pTr} \downarrow \rho \alpha. \downarrow (r \varrho) x$. For *completed* trace equivalence this is the case [24], but we can find a simple counter-example for p -trace equivalence.

Example 5.1. Let $x \in N$ be the process $a. \downarrow (1 \cdot \frac{1}{2}) b. \downarrow \frac{1}{2} c. 1\delta$, and $y \in N$ the process $a. \downarrow 1b. \downarrow (\frac{1}{2} \cdot \frac{1}{2}) c. 1\delta$. Then we have that

- $pTr[x]$ is $\{(\varepsilon, 1), (a, 1), (ab, \frac{1}{2}), (abc, \frac{1}{4})\}$, and
- $pTr[y]$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{4})\}$.

It is clear these processes are not p -trace equivalent.

The problem is that the shorter traces are not respected by this transformation. The transformation is only valid when both instances of the shorter trace are already present in the process. We can instead move a junction to another part of the process where some of the probabilities need to be adapted. Let us give an example.

Example 5.2. Let $x \in N$ be the process $a. \downarrow (1 \cdot \frac{1}{2}) b. \downarrow \frac{1}{2} c. 1\delta + a. \downarrow 1b. 1\delta$, and $y \in N$ be the process $a. \downarrow (1 \cdot \frac{1}{2}) b. 1\delta + a. \downarrow 1b. \downarrow (\frac{1}{2} \cdot \frac{1}{2}) c. 1\delta$, as shown respectively in Figure 7, then we have that

- $pTr[x]$ and $pTr[y]$ are equal to $\{(\varepsilon, 1), (a, 1), (ab, \frac{1}{2}), (ab, 1), (abc, \frac{1}{4})\}$.

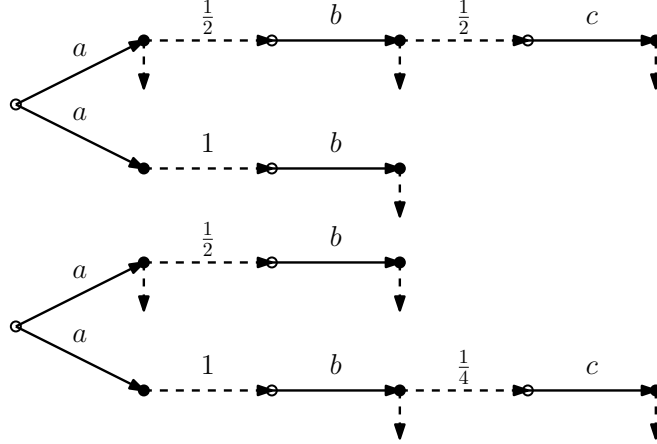


Figure 7: The two transition systems in Example 5.2

Hence, the processes are p -trace equivalent.

Inspired by the example we can formulate axiom PT2. This gives us a complete axiomatisation for processes where every probabilistic composition is a junction. We call such process a *street*, and we will see that any probabilistic process can be converted into a street under p -trace equivalence using some additional axioms.

5.3 Axioms for Distribution

We showed that, for p -trace equivalence, alternative composition distributes over junctions (PT1). We wonder if the same holds for more complex probabilistic compositions. In general, we ask ourselves whether

$$\alpha. \sum_{i \in I} \rho_i (x_i + y_i) =_{pTr} \alpha. \sum_{i \in I} \rho_i x_i + \alpha. \sum_{i \in I} \rho_i y_i$$

for sequences of processes $x, y \in N^+$ and probabilities $\rho \in \mathcal{P}^+$. We can show by a counter-example that this is not the case.

Example 5.3. Let $x \in N$ be the process

$$a. \left[\frac{1}{2} \left(b. \downarrow_{\frac{1}{8}} c.1\delta + b. \downarrow_{\frac{3}{8}} c.1\delta \right) \oplus \frac{1}{2} \left(b. \downarrow_{\frac{1}{8}} c.1\delta + b. \downarrow_{\frac{5}{8}} c.1\delta \right) \right],$$

and $y \in N$ be the process

$$a. \left(\frac{1}{2}b. \downarrow \frac{1}{8}c.1\delta \oplus \frac{1}{2}b. \downarrow \frac{1}{8}c.1\delta \right) + \alpha. \left(\frac{1}{2}b. \downarrow \frac{3}{8}c.1\delta \oplus \frac{1}{2}b. \downarrow \frac{5}{8}c.1\delta \right),$$

as shown respectively in Figures 4 and 8.

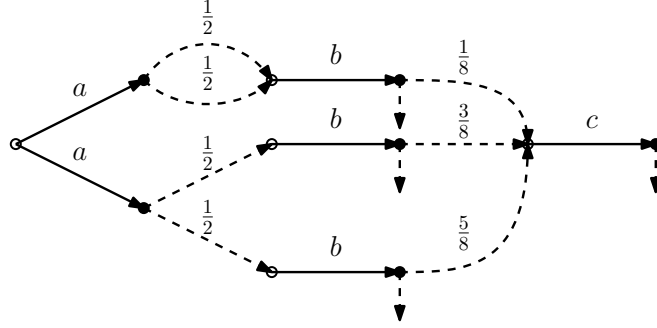


Figure 8: The process $a. \left(\frac{1}{2}b. \frac{1}{8}c.1\delta \oplus \frac{1}{2}b. \frac{1}{8}c.1\delta \right) + \alpha. \left(\frac{1}{2}b. \frac{3}{8}c.1\delta \oplus \frac{1}{2}b. \frac{5}{8}c.1\delta \right)$

We have that

- $pTr[x]$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{8}), (abc, \frac{2}{8}), (abc, \frac{3}{8}), (abc, \frac{4}{8})\}$,
- $pTr[y]$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{8}), (abc, \frac{4}{8})\}$.

From this we can conclude that $pTr[x] \supset pTr[y]$, thus the processes are not p -trace equivalent.

The processes in Example 5.3 are not equivalent because their traces $(x_1 \cup y_1) \uplus (x_2 \cup y_2)$ are not equal to $(x_1 \uplus x_2) \cup (y_1 \uplus y_2)$. This would only have been the case if the traces of x_i and y_i did not share the same action sequences. We call two sets of traces that do not have a single action sequence in common *disjoint*. We extend our notion of disjointness to processes.

Definition 5.4. We call two processes or process expressions disjoint when their leading (the first following) actions form two disjoint sets. We call two *sets* of processes disjoint when their processes are pairwise disjoint.

We find that for disjoint sets of processes, alternative composition distributes over probabilistic composition (PD1).

The notion of disjoint processes we have given is simpler, and therefore more restrictive than the notion of disjoint p -trace sets. We can, however, find

disjoint behaviour deeper in the process. The processes $a.\downarrow_{\frac{1}{2}}b.1\delta + a.\downarrow_{\frac{1}{3}}b.1\delta$ and $a.\downarrow_{\frac{1}{2}}c.1\delta$ show disjoint behaviour, even though they both have leading actions $\{a\}$. The processes become distinct at a deeper level. We generalise this: the process $\alpha_1.\downarrow_{\rho_1}\cdots\alpha_n.\downarrow_{\rho_n}x$ shows disjoint behaviour to process $\alpha_1.\downarrow_{\varrho_1}\cdots\alpha_n.\downarrow_{\varrho_n}y$ when x and y are disjoint, where $\alpha \in \mathcal{A}^+$ is a sequence of actions, $\rho, \varrho \in \mathcal{P}^+$ are sequences of probabilities, and $x, y \in N$ are processes. We are interested if distribution like PD1 also holds for processes with disjoint behaviour. This is the case but there is a catch. We must respect the part that is not disjoint. Therefore the distribution results in three terms: the two disjoint terms, and a term that remains undistributed, which includes the processes until the part they become disjoint.

Example 5.5. Let $x_1, x_2 \in N$ be the processes $b.\downarrow_{\frac{1}{8}}c.1d.1\delta$ and $b.\downarrow_{\frac{1}{8}}c.1d.1\delta$, and let $y_1, y_2 \in N$ be the processes $b.\downarrow_{\frac{3}{8}}c.1e.1\delta$ and $b.\downarrow_{\frac{5}{8}}c.1e.1\delta$. Note that x and y show disjoint behaviour. Then we have that

- $pTr \left[a. \left[\frac{1}{2} (x_1 + y_1) \oplus \frac{1}{2} (x_2 + y_2) \right] \right]$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{8}), (abc, \frac{2}{8}), (abc, \frac{3}{8}), (abc, \frac{4}{8}), (abcd, \frac{1}{8}), (abce, \frac{4}{8})\}$, which is also
- $pTr \left[a. \left(\frac{1}{2}x_1 \oplus \frac{1}{2}x_2 \right) + a. \left(\frac{1}{2}y_1 \oplus \frac{1}{2}y_2 \right) + a. \left[\frac{1}{2} \left(b.\downarrow_{\frac{1}{8}}c.1\delta + b.\downarrow_{\frac{3}{8}}c.1\delta \right) \oplus \frac{1}{2} \left(b.\downarrow_{\frac{1}{8}}c.1\delta + b.\downarrow_{\frac{5}{8}}c.1\delta \right) \right] \right]$.

We can find axioms for disjoint behaviour at any finite depth, but for completeness, we only need to consider the quantified versions of level 1 (PD2) and level 2 (PD3).

5.4 Axioms for Unification

We have already seen that δ terms in a probabilistic composition can be unified (PA3). We wonder if we can also unify more complex terms. We consider $\rho x \oplus \varrho x = (\rho + \varrho)x$ and $\rho x \oplus \varrho y = (\rho + \varrho)(x + y)$, and present counter-examples.

Example 5.6. Let $x \in N$ be the process $b.\downarrow_{\frac{1}{2}}c.1\delta + b.1c.1\delta$, and $\rho, \varrho \in \mathcal{P}$ be probabilities both $\frac{1}{2}$. We see that

- $pTr \left[a. \left(\frac{1}{2}x \oplus \frac{1}{2}x \right) \right]$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{2}), (abc, \frac{3}{4}), (abc, 1)\}$, and

- $pTr \llbracket a.(\frac{1}{2} + \frac{1}{2})x \rrbracket$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{2}), (abc, 1)\}$, as shown in Figure 6.

Note that the trace $(abc, \frac{3}{4})$ is not present in both p -trace sets, these processes are not p -trace equivalent.

Example 5.7. Let $x \in N$ be the process $b.1\delta$, $y \in N$ be the process $c.1\delta$, and $\rho, \varrho \in \mathcal{P}$ be the probabilities $\frac{1}{3}$ and $\frac{2}{3}$. Note that x and y are disjoint. Then we have that

- $pTr \llbracket a.(\frac{1}{3}x \oplus \frac{2}{3}y) \rrbracket$ is $\{(\varepsilon, 1), (a, 1), (ab, \frac{1}{3}), (ac, \frac{2}{3})\}$, and
- $pTr \llbracket a.(\frac{1}{3} + \frac{2}{3})(x + y) \rrbracket$ is $\{(\varepsilon, 1), (a, 1), (ab, 1), (ac, 1)\}$.

It is clear that these processes are not p -trace equivalent.

From the last example it becomes apparent that we can not unify terms that are disjoint. The same set of leading actions need to be present. Example 5.6 shows that, even when this is the case, alternative composition inside terms causes problems. However, for the simple case where both terms have just a sequential composition, with the same action, the terms can be unified (PU1). The result is a new probabilistic composition, which is the combination of both terms, each multiplied by a correction factor, so their probabilities add up to 1.

In the previous section we talked about disjoint behaviour. We are interested if we can apply this also to terms in a probabilistic composition. This appears to be the case, although it manifests itself in a different way. When unifying two groups of processes with disjoint behaviour the result is a Cartesian product of the groups. This is analogous to the way the \uplus -operator sums probabilities with equal action sequences.

Example 5.8. Let $u_1, u_2 \in P(1)$ be the process expressions $\downarrow \frac{1}{8}c.1d.1\delta$ and $\downarrow \frac{3}{8}c.1d.1\delta$, and let $v_1, v_2 \in P(1)$ be the process expressions $\downarrow \frac{1}{8}c.1e.1\delta$ and $\downarrow \frac{5}{8}c.1e.1\delta$. Note that u and v show disjoint behaviour. Then we have that

- $pTr \llbracket a. [\frac{1}{2}(b.u_1 + b.u_2) \oplus \frac{1}{2}(b.v_1 + b.v_2)] \rrbracket$ is
$$\left\{ \begin{array}{l} (\varepsilon, 1), (a, 1), (ab, 1), (abc, \frac{1}{8}), (abc, \frac{2}{8}), (abc, \frac{3}{8}), (abc, \frac{4}{8}), \\ (abcd, \frac{1}{16}), (abcd, \frac{3}{16}), (abce, \frac{1}{16}), (abce, \frac{5}{16}) \end{array} \right\},$$

which is also

$$pTr \llbracket a.1 (b. [\frac{1}{2} \cdot u_1 \oplus \frac{1}{2} \cdot v_1] + b. [\frac{1}{2} \cdot u_1 \oplus \frac{1}{2} \cdot v_2] + b. [\frac{1}{2} \cdot u_2 \oplus \frac{1}{2} \cdot v_1] + b. [\frac{1}{2} \cdot u_2 \oplus \frac{1}{2} \cdot v_2]) \rrbracket.$$

The processes are p -trace equivalent.

Like distribution, we can find unification axioms for disjoint behaviour at any finite depth, but for completeness, we only require the versions of level 1 (PU2) and level 2 (PU3).

5.5 Auxiliary Axioms

We also present a couple of auxiliary axioms that are not required for completeness. They can be derived from the other axioms.

First, we present a special case of Axiom PU3. Suppose that both process expression groups U and V contain only δ . We can simply sum up their probabilities (X1). We present this axiom for convenience as it will be used as a base case in the completeness proof.

We also present an axiom that, although it can be directly derived from Axiom PU1, is not intuitively clear. We can split off the δ -part of a junction and distribute it over a preceding probabilistic composition (X2). This is a noteworthy trait, but we will not further employ it.

6 Soundness and Completeness

In the previous section, we presented axioms summarised in Table 3. All axioms are sound, of which the extensive proofs can be found in [26]. Here we concentrate to prove that the axioms form a complete axiomatisation. We begin by defining a subset of probabilistic processes, where each probabilistic composition can be written as a junction. We call these processes *streets*. We show that any process can be rewritten as a street under p -trace equivalence.

We also define a subset of streets, processes with only deadlocks, sequential compositions, and junctions. We call these processes *lanes*. We show that any street can be rewritten as a *sum* of lanes under p -trace equivalence. We also show a number of interesting properties that relate lanes to p -traces. Finally, these properties are used to define a normal form, from which completeness is proved.

6.1 Streets

Definition 6.1. We define the set of process expressions $Street \subseteq (N \cup \mathbf{P})$ as the minimal set, where $x, y \in N$, $u \in P(1)$, $\rho \in \mathcal{P}$, and $\alpha \in \mathcal{A}$, satisfying that

- δ and 1δ are streets,
- if x and y are streets, then so is $x + y$,
- if u is a street, then so is $\alpha.u$, and
- if x is a street other than δ , then so is $\downarrow\rho x$.

We defined streets informally as a process where each probabilistic composition is a junction. With this statement in mind, we can then deduce from the syntax that every street can be written as a sum of action prefixed junctions. Using Axioms A1-4 we can generalise this to a summation over a set. We formalise this.

Lemma 6.2. Let $x \in (N \cap Street)$ be a street process. There is a sequence $S \in (\mathcal{A} \times \mathcal{P} \times Street)^*$ such that $x = \sum_{(\alpha, \rho, y) \in S} \alpha.\downarrow\rho y$.

We also consider partial streets, which are processes where all but their first probabilistic composition is a junction. In particular, we look at processes of shape $\omega.\sum_{i \in I} \rho_i x_i$, where $\omega \in \mathcal{A}$ is an action, $\rho_i \in \mathcal{P}$ are probabilities, and $x_i \in (N \cap Street)$ are street processes.

We can generalise Axiom PA3 for such processes (Lemma 6.4).

Remark 6.3. For simplicity, we will also use the notation of $\downarrow\sum_{i \in I} u_i$ for I is empty as an abbreviation for 1δ .

Lemma 6.4. Let $\omega.\sum_{i \in I} \rho_i x_i$ be a partial street. This street is equivalent to a process of form $\omega.\downarrow\sum_{i \in I | x_i \neq \delta} \rho_i x_i$.

Proof: In case there are no processes x_i that are δ , it is intrinsically true. In case all processes x_i are δ we have that $\omega.\sum_{i \in I} \rho_i x_i = \omega.\sum_{i \in I} \rho_i \delta = \omega.1\delta$ (PA3) for which we will write $\omega.\downarrow\sum_{i \in I | x_i \neq \delta} \rho_i x_i$.

Otherwise, we have that

$$\begin{aligned}
\omega. \sum_{i \in I} \rho_i x_i &= \omega. \left(\sum_{i \in I | x_i \neq \delta} \rho_i x_i \oplus \sum_{i \in I | x_i = \delta} \rho_i x_i \right) && \text{split domain} \\
&= \omega. \left(\sum_{i \in I | x_i \neq \delta} \rho_i x_i \oplus \sum_{i \in I | x_i = \delta} \rho_i \delta \right) \\
&= \omega. \left(\sum_{i \in I | x_i \neq \delta} \rho_i x_i \oplus \left(\sum_{i \in I | x_i = \delta} \rho_i \right) \delta \right) && \text{PA3} \\
&= \omega. \downarrow \sum_{i \in I | x_i \neq \delta} \rho_i x_i && \text{definition of } \downarrow
\end{aligned}$$

□

Remark 6.5. In the following lemmas, we make extensive use of Lemma 6.2 to unfold street processes. In every case, we denote the resulting sequence by an upper-case variant of the variable. For the sake of brevity, we do not mention this explicitly.

More interestingly, we can split a partial street into a sum, where each street starts with the same action.

Lemma 6.6. Let $\omega. \sum_{i \in I} \rho_i x_i$ be a partial street. It is equivalent to a process of form

$$\sum_{\alpha \in \mathcal{A}} \omega. \sum_{i \in I} \rho_i \sum_{(\varrho, y) \in X_i^\alpha} \alpha. \downarrow \varrho y,$$

where $X_i^\alpha = \{(\varrho, y) \mid (\alpha, \varrho, y) \in X_i\}$.

Proof: We show that

$$\begin{aligned}
\omega. \sum_{i \in I} \rho_i x_i &= \omega. \sum_{i \in I} \rho_i \sum_{(\alpha, \varrho, y) \in X_i} \alpha. \downarrow \varrho y && \text{Lemma 6.2, Remark 6.5} \\
&= \omega. \sum_{i \in I} \rho_i \sum_{\alpha \in \mathcal{A}} \sum_{(\varrho, y) \in X_i^\alpha} \alpha. \downarrow \varrho y && \text{A1, A2, A4} \\
&= \sum_{\alpha \in \mathcal{A}} \omega. \sum_{i \in I} \rho_i \sum_{(\varrho, y) \in X_i^\alpha} \alpha. \downarrow \varrho y.
\end{aligned}$$

PD1 (definition of disjoint processes) □

We can repeat this process one level deeper. We split a partial street into a sum, where each street starts with the same *two* actions in a row.

Lemma 6.7. Let $\omega. \sum_{i \in I} \rho_i x_i$ be a partial street. Then, it is equivalent to a process of form

$$\sum_{\alpha \in \mathcal{A}, \beta \in \mathcal{A}} \omega. \downarrow \sum_{i \in I | X_i^\alpha \neq []} \rho_i \sum_{(\varrho, y) \in X_i^\alpha} \alpha. \downarrow \varrho \sum_{(p, z) \in Y^\beta} \beta. \downarrow pz,$$

where $X_i^\alpha = \{(\varrho, y) \mid (\alpha, \varrho, y) \in X_i\}$ and similarly is Y^β .

We generalised Axiom PA3 for partial streets in Lemma 6.4. This allows us to unify terms with empty streets. We continue this one level deeper, and unify terms with streets of length 1 that all start with a particular action, let us say α . Assume there is at least one street $\alpha. \downarrow x$ in a partial street, where $x \in N$ is a process other than δ . We can unify all terms $\alpha.1\delta$ with longer counterparts. That means that the process $a. \downarrow [\frac{1}{3}(b.1c.1\delta + b.1\delta) \oplus \frac{1}{3}b.1\delta]$ can be rewritten to $a. \downarrow \frac{2}{3}b. \downarrow \frac{1}{2}c.1\delta$.

Lemma 6.8. Let $\sum_{i \in I} \rho_i \sum_{(\varrho, y) \in X_i} \alpha. \downarrow \varrho y$ be a partial street, where $\alpha \in \mathcal{A}$ is an action, and for some i there is at least one y that is not δ . This street can be rewritten to a partial street $\sum_{i \in I} \rho'_i \sum_{(\varrho, y) \in X'_i} \alpha. \downarrow \varrho y$, for which it holds that all values of X'_i and y are neither empty nor δ .

We can continue this one level deeper and unify streets of length 2. Instead, we show that we can split up a process into two parts. Let's assume we have a partial street where each term is a street of at least length 2, having the same two consecutive actions, call them α and β . We can then group all streets of exactly length 2 into one half, and group the streets that are longer into another. Furthermore, for the latter group, we can unify their probabilities in each term so that they are all of the form $\alpha. \downarrow \hat{\rho} \beta. \downarrow \varrho x$, where $\hat{\rho}$ is a particular probability.

First, we show that we can separate a particular length-2 street from a sum.

Lemma 6.9. Let $\sum_{i \in I} \rho_i \sum_{(\varrho, y) \in X_i} \alpha. \downarrow \varrho \sum_{(p, z) \in Y} \beta. \downarrow pz$ be a partial street, where $\alpha, \beta \in \mathcal{A}$ are actions, and all X_i and Y are non-empty. Then, it is equivalent to a process of form

$$\left(\begin{array}{l} \omega. \downarrow \sum_{i \in I} \rho_i \sum_{(\varrho, y) \in X_i} \alpha. \downarrow \varrho \beta. 1\delta \\ \omega. \downarrow \sum_{i \in I} \rho_i \sum_{(\varrho, y) \in X_i} \sum_{(p, z) \in Y} \alpha. \downarrow \hat{\varrho}_i \beta. \downarrow \frac{\varrho p}{\hat{\varrho}_i} z \end{array} \right) +$$

where $\hat{\varrho}_i$ is a particular probability depending on i .

We now have the building blocks to show that we can rewrite any process to a street under p -trace equivalence.

Lemma 6.10. Let $s = \omega. \sum_{i \in I} \rho_i x_i$ be a partial street. There exists a street $s' \in (N \cap Street)$ equivalent to s .

Proof: We have that $\omega. \sum_{i \in I} \rho_i x_i = \sum_{\alpha \in \mathcal{A}, \beta \in \mathcal{A}} \omega. \downarrow \sum_{i \in I | X_i^\alpha \neq []} \rho_i \sum_{(q,y) \in X_i^\alpha} \alpha. \downarrow \varrho \sum_{(p,z) \in Y^\beta} \beta. \downarrow pz$ by Lemma 6.7. We prove for each of the terms of the leading sum separately that it can be rewritten to a street. Let $\alpha, \beta \in \mathcal{A}$ be the actions of a particular term. In case that all instance of X_i^α are empty we have the term $\omega. \downarrow \sum_{i \in I | X_i^\alpha \neq []} = \omega. 1\delta$, which already is a street. In case all instances of Y^β are empty we have that

$$\begin{aligned} \omega. \downarrow \sum_{i \in I | X_i^\alpha \neq []} \rho_i \sum_{(q,y) \in X_i^\alpha} \alpha. \downarrow \varrho \sum_{(p,z) \in Y^\beta} \beta. \downarrow pz &= \omega. \downarrow \sum_{i \in I | X_i^\alpha \neq []} \rho_i \sum_{(q,y) \in X_i^\alpha} \alpha. 1\delta \\ &= \omega. \downarrow \sum_{i \in I | X_i^\alpha \neq []} \rho_i \alpha. 1\delta \quad \text{A3} \\ &= \omega. \downarrow \left(\sum_{i \in I | X_i^\alpha \neq []} \rho_i \right) \alpha. 1\delta \\ &\quad \text{gen. PU1,} \end{aligned}$$

results in a street process.

Otherwise, at least one instance of Y^β is non-empty. We can rewrite this term to

$$\omega. \downarrow \sum_{j \in J} \rho'_j \sum_{(q,y) \in X_j'^\alpha} \alpha. \downarrow \varrho \sum_{(p,z) \in Y^\beta} \beta. \downarrow pz$$

for some ρ' and X'^α , such that all instances of X'^α and Y^β are non-empty (Lemma 6.8). This allows us to rewrite the term to

$$\left(\begin{array}{l} \omega. \downarrow \sum_{j \in J} \rho'_j \sum_{(q,y) \in X_j'^\alpha} \alpha. \downarrow \varrho \beta. 1\delta \\ \omega. \downarrow \sum_{j \in J} \rho'_j \sum_{(q,y) \in X_j'^\alpha} \sum_{(p,z) \in Y^\beta} \alpha. \downarrow \hat{\varrho}_j \beta. \downarrow \frac{\varrho p}{\hat{\varrho}_j} z \end{array} \right) +$$

We prove this for both terms separately.

Let ρ' be the sum of probabilities $(\sum_{j \in J} \rho'_j)$. The first term

$$\begin{aligned} & \omega.\downarrow \sum_{j \in J} \rho'_j \sum_{(e,y) \in X_j'^\alpha} \alpha.\downarrow \varrho \beta.1\delta \\ &= \omega.\downarrow \rho' \sum_{(\varrho_1, -) \in X_1'^\alpha, \dots, (\varrho_n, -) \in X_n'^\alpha} \alpha.\downarrow \frac{(\sum_{j \in J} \rho'_j \varrho_j)}{\rho'} \beta.1\delta \quad \text{generalized X1} \end{aligned}$$

results in a street process.

The other term

$$\begin{aligned} & \omega.\downarrow \sum_{j \in J} \rho'_j \sum_{(e,y) \in X_j'^\alpha} \sum_{(p,z) \in Y^\beta} \alpha.\downarrow \hat{\varrho}_j \beta.\downarrow \frac{\varrho p}{\hat{\varrho}_j} z \\ &= \omega.\downarrow \sum_{j \in J} \rho'_j \alpha.\downarrow \hat{\varrho}_j \sum_{(e,y) \in X_j'^\alpha} \sum_{(p,z) \in Y^\beta} \beta.\downarrow \frac{\varrho p}{\hat{\varrho}_j} z \quad \text{PT1 } (X_j'^\alpha, Y^\beta \neq \square) \\ &= \omega.\downarrow \rho' \alpha.\downarrow \sum_{j \in J} \frac{\rho'_j \hat{\varrho}_j}{\rho'} \sum_{(e,y) \in X_j'^\alpha} \sum_{(p,z) \in Y^\beta} \beta.\downarrow \frac{\varrho p}{\hat{\varrho}_j} z \quad \text{generalized PU1} \\ &= \omega.\downarrow \rho' t \text{ where } t := \alpha.\downarrow \sum_{j \in J} \frac{\rho'_j \hat{\varrho}_j}{\rho'} \sum_{(e,y) \in X_j'^\alpha} \sum_{(p,z) \in Y^\beta} \beta.\downarrow \frac{\varrho p}{\hat{\varrho}_j} z \text{ note that } t \in N \\ &= \omega.\downarrow \rho' t' \quad \text{induction on } t, \text{ where } t' \text{ is a street process} \end{aligned}$$

results also in a street process.

Since a sum of streets is also a street itself we have now shown we can rewrite s to a street process. Hence, the lemma is valid. \square

Theorem 6.11. For every probabilistic process there is an equivalent street process, derivable using the axioms of Table 3.

Proof: Let $x \in N$ be a probabilistic process. We prove by induction on the structure that we can rewrite x to a street process.

- Suppose that $x = \delta$. In this case it already is a street.
- Suppose that $x = y + z$, where $y, z \in N$ are processes. Then, by induction, we can rewrite y and z to street processes y' and z' . By substitution, we find $y' + z'$, which is a street.

- Suppose that $x = \omega.u$, where $\omega \in \mathcal{A}$ is an action and $u \in P(1)$ is a process expression. We have a sequence $[(\rho_1, y_1), \dots, (\rho_n, y_n)]$ such that $u = \sum_{i \in I} \rho_i y_i$ (Corollary 2.5). By induction, we can rewrite each process y_i to a street y'_i . By substitution, we find $\omega. \sum_{i \in I} \rho_i y'_i$ which we can rewrite to a street (Lemma 6.10).

In conclusion, we have shown that all processes $x \in N$ we can rewrite to a street process, under p -trace equivalence. \square

6.2 Lanes

In the previous section, we introduced streets and showed we can rewrite any process to a street under p -trace equivalence. In this section we introduce a more restricted group of processes called *lanes*. In lanes we eliminate every form of branching, except junctions. Lanes are completely linear processes. We show that any street, and by extension any probabilistic process, can be rewritten to a sum of lanes, which is an alternative composition where each term is a lane, under p -trace equivalence.

Definition 6.12. We define the set of process expressions $Lane \subseteq (N \cup \mathbf{P})$ as the minimal set, where $x \in N$, $u \in P(1)$, $\rho \in \mathcal{P}$, and $\alpha \in \mathcal{A}$, satisfying that

- δ and 1δ are lanes,
- if u is a lane, then so is $\alpha.u$, and
- if x is a lane other than δ , then so is $\downarrow \rho x$.

Theorem 6.13. For every street process there is an equivalent sum of lanes, derivable using the axioms of Table 3.

Proof: Let $x \in (N \cap Street)$ be a street process. We prove by induction on the structure that we can rewrite x to a sum of lanes.

- Suppose that $x = \delta$. It already is a sum of lanes, namely, the empty sum.
- We have never that $x = 1\delta$, since $1\delta \notin N$.
- Suppose that $x = y + z$, where $y, z \in N$ are processes. Note that by definition y and z are also streets. By induction, we can rewrite them to a sum of lanes y' and z' . By substitution, we find $y' + z'$, which is also a sum of lanes, by domain union.

- Suppose that $x = \alpha.u$, where $\alpha \in \mathcal{A}$ is an action and $u \in P(1)$ is a process expression. There are two cases.
 - Suppose that $u = 1\delta$, then x is already a sum of lanes, namely, the sum with only the term $\alpha.1\delta$, which is a lane.
 - Suppose that $u = \downarrow\rho y$, where $\rho \in \mathcal{P}$ is a probability and $y \in N$ is a process. Note that by definition y is also a street. By induction, we can rewrite it to a sum of lanes y' . By substitution we find $\alpha.\downarrow\rho y'$. If y' is empty we have that $\alpha.\downarrow\rho\delta = \alpha.1\delta$ (PA3) which we already saw is a sum of lanes. Otherwise, we have that $\alpha.\downarrow\rho y' = \alpha.\downarrow\rho \sum_{z \in Y'} z$ with a suitable Y' , where all instances of z are lanes. Since we assumed Y' is not empty, we can rewrite this to $\sum_{z \in Y'} \alpha.\downarrow\rho z$ (PT1), which is a sum of lanes.

Together, this proves the theorem. □

We are interested if we can find a relationship between lanes and p -traces, and find interesting properties of lanes regarding their p -trace set. One notable property is that the p -traces of a lane are unique to that lane. In other words, no two lanes exist with the same p -trace set unless they are equal by construction.

Lemma 6.14. Lane processes or expressions that are p -trace equivalent are also syntactically equal.

When we look at a p -trace set associated with a lane, one thing immediately stands out. Each trace has a unique action sequence. We formalise this.

Lemma 6.15. Let s be a lane and let (σ, ρ) and (σ, ϱ) be two traces of $pTr\llbracket s \rrbracket$. Then $(\sigma, \rho) = (\sigma, \varrho)$.

We are interested in the trace with the longest action sequence in a p -trace set associated with a lane. We can conclude from the last lemma that there is only one such trace. We call this the *characteristic* trace of a lane. Since by construction the p -trace set is never empty we know this trace must always exist. For a trace shorter than the characteristic lane, let's say (σ, ρ) , we can also find a shorter lane so that this lane has a characteristic trace equal to (σ, ρ) . We show that we can rewrite any lane to a sum of lanes so that each lane in this sum corresponds to a trace in the original lane.

Definition 6.16. Let $s \in \text{Lane}$ be a lane and let (σ, ρ) be a trace in $pTr[[s]]$. We call (σ, ρ) the characteristic trace of s iff for all traces $(\varsigma, \varrho) \in pTr[[s]]$ it holds that ς is a prefix of σ ($\varsigma \leq \sigma$). We denote this trace as $\mathcal{C}(s)$.

Lemma 6.17. Let x be a lane process. There is a sum of lanes $s = \sum_{t \in S} t$ equivalent to x , such that $\bigcup_{t \in S} \{\mathcal{C}(t)\} = pTr[[x]]$.

Proof: We prove by induction on the structure that we can rewrite x to such s .

- Suppose that $x = \delta$. Then $pTr[[x]] = \{(\varepsilon, 1)\}$. Hence, x intrinsically satisfies the condition.
- Suppose that $x = \alpha.1\delta$. Then $pTr[[x]] = \{(\varepsilon, 1), (\alpha, 1)\}$. We can rewrite x to $\alpha.1\delta + \delta$ (A4) which satisfies the condition.
- Suppose that $x = \alpha.\downarrow\rho y$, where $y \in N$ is a lane. Then $pTr[[x]] = \alpha.\rho \cdot pTr[[y]]_{\bar{\varepsilon}} \cup \{(\varepsilon, 1), (\alpha, 1)\}$. By induction we can rewrite y to a sum of lanes $\sum_{t \in Y} t$ for which the condition holds. Hence, we can rewrite x to

$$\begin{aligned}
\alpha.\downarrow\rho y &= \alpha.\downarrow\rho \sum_{t \in Y} t && \text{induction} \\
&= \alpha.\downarrow\rho \left(\sum_{t \in Y} t + \delta \right) && \text{A4} \\
&= \sum_{t \in Y} \alpha.\downarrow\rho t + \alpha.1\delta + \delta. && \text{PT1, PA3, A4}
\end{aligned}$$

Note that

$$\begin{aligned}
&\bigcup_{t \in Y} \{\mathcal{C}(\alpha.\downarrow\rho t)\} \cup \{\mathcal{C}(\alpha.1\delta)\} \cup \{\mathcal{C}(\delta)\} \\
&= \bigcup_{t \in Y} \{\mathcal{C}(\alpha.\downarrow\rho t)\} \cup \{(\varepsilon, 1), (\alpha, 1)\} \\
&= \alpha.\rho \cdot \bigcup_{t \in Y} \{\mathcal{C}(t)\}_{\bar{\varepsilon}} \cup \{(\varepsilon, 1), (\alpha, 1)\} \\
&= \alpha.\rho \cdot pTr[[y]]_{\bar{\varepsilon}} \cup \{(\varepsilon, 1), (\alpha, 1)\} && \text{IH,}
\end{aligned}$$

which is $pTr[[x]]$. Thus, the condition is satisfied.

This means that we have shown the lemma holds. \square

If we look at a sum of lanes we see that different lanes could have the same characteristic trace. For instance in the process $a.1b.\downarrow\frac{1}{4}c.1\delta + a.\downarrow\frac{1}{2}b.\downarrow\frac{1}{2}c.1\delta$ we have two lanes both with characteristic trace $(abc, \frac{1}{4})$. Note that the probabilities drop off sooner for the second lane, in comparison to the first one. Conversely, all traces corresponding to the first lane have generally larger probabilities compared to the second lane. We can aggregate this. We can distinguish lanes, in the context of a sum of lanes, whose corresponding traces have the largest probabilities, amongst a group of lanes with the same characteristic trace. We call such lanes *principal*. We will show that principal lanes that are part of the same sum, with the same characteristic trace, are syntactically equal to each other. We also show that we can rewrite a sum of lanes to a sum of *principal* lanes.

Definition 6.18. Let $s = \sum_{t \in S} t$ be a sum of lanes, and let $x \in S$ be a lane part of s . We call x principal iff for all traces $(\sigma, \rho) \in (pTr[x] \setminus \{\mathcal{C}(x)\})$ and all traces $(\sigma, \varrho) \in pTr[s]$ it holds that $\rho \geq \varrho$. We call s the *context* of lane x .

Lemma 6.19. Let $s = \sum_{t \in S} t$ be a sum of lanes, and let $x, y \in S$ be two principal lanes part of s such that $\mathcal{C}(x) = \mathcal{C}(y)$. Then $x = y$.

Proof: First, we prove that $pTr[x] = pTr[y]$. We just show that $pTr[x] \subseteq pTr[y]$ since the proof for the converse is symmetrical.

Let (σ, ρ) be a trace in $pTr[x]$. We have two cases.

- Suppose that $(\sigma, \rho) = \mathcal{C}(x)$. Then we have that $(\sigma, \rho) \in pTr[y]$, since $\mathcal{C}(y) \in pTr[y]$ by definition and $\mathcal{C}(x) = \mathcal{C}(y)$.
- Suppose that $(\sigma, \rho) \neq \mathcal{C}(x)$. Then there must be a trace $(\sigma, \varrho) \in pTr[y]$, since a longer trace $\mathcal{C}(x) = \mathcal{C}(y)$ exists in $pTr[y]$. Because both lanes are principal, we have that $\rho \geq \varrho$ and $\varrho \geq \rho$, and thus $\rho = \varrho$. We conclude that $(\sigma, \rho) \in pTr[y]$.

Together we have that $pTr[x] = pTr[y]$, which means that $x = y$ by Lemma 6.14. \square

Lemma 6.20. Let $s = \sum_{t \in S} t$ be a sum of lanes. There is a sum of *principal* lanes equivalent to s .

Proof: We prove this lemma by induction on the structure of s . First note that any lane δ , $\alpha.1\delta$ or $\alpha.\downarrow\rho\beta.1\delta$, where $\alpha, \beta \in \mathcal{A}$ are actions and $\rho \in \mathcal{P}$ is

a probability, is intrinsically a principal lane. For the remainder of the proof we can assume that we only need to prove the lemma for lanes of length larger than 2 (*).

Note that we can split up S into sequences having the same two consecutive actions (A1, A2). Let $S^{\alpha\beta}$ be $\left[t \mid t \in S \wedge \exists \rho \in \mathcal{P}, u \in \mathbf{P} t = \alpha.\downarrow\rho\beta.u \right]$. Since these sequences are disjoint we can prove the lemma for each one separately. If we prove for arbitrary actions $\alpha, \beta \in \mathcal{A}$ that we can rewrite $s^{\alpha\beta} = \sum_{t \in S^{\alpha\beta}} t$ to a sum of principal lanes, then by extension the same holds for s .

By assumption of (*) we know that $S^{\alpha\beta}$ is not empty. So, we can find a probability $\hat{\rho} \in \mathcal{P}$ such that for all lanes $\alpha.\downarrow\rho\beta.u \in S^{\alpha\beta}$ it holds that $\hat{\rho} \geq \rho$. Then we can rewrite $s^{\alpha\beta}$ to

$$\begin{aligned}
\sum_{t \in S^{\alpha\beta}} t &= \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\rho\beta.\downarrow qx && \text{for some } \rho, \varrho \in \mathcal{P} \text{ and } x \in N, (*) \\
&= \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\rho\beta.\downarrow qx + \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\hat{\rho}\beta.1\delta && \text{A4, PD2} \\
&= \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\rho\beta.1\delta + \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\hat{\rho}\beta.\downarrow\frac{\rho\varrho}{\hat{\rho}}x && \text{PT2} \\
&= \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\rho\beta.1\delta + \alpha.\downarrow\hat{\rho} \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \beta.\downarrow\frac{\rho\varrho}{\hat{\rho}}x && \\
&&& \text{PT1 } (S^{\alpha\beta} \neq \emptyset) \\
&= \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\rho\beta.1\delta + \alpha.\downarrow\hat{\rho} \sum_{\alpha.\downarrow\rho\beta.\downarrow qx' \in S'^{\alpha\beta}} \beta.\downarrow\frac{\rho\varrho}{\hat{\rho}}x' && \\
&&& \text{induction, where all elements of } S'^{\alpha\beta} \text{ are principal lanes} \\
&= \sum_{\alpha.\downarrow\rho\beta.\downarrow qx \in S^{\alpha\beta}} \alpha.\downarrow\rho\beta.1\delta + \sum_{\alpha.\downarrow\rho\beta.\downarrow qx' \in S'^{\alpha\beta}} \alpha.\downarrow\hat{\rho}\beta.\downarrow\frac{\rho\varrho}{\hat{\rho}}x'. && \text{PT1}
\end{aligned}$$

Note that the left sum is intrinsically a sum of principal lanes. For the right sum we have that the first junction is maximal by construction, and the following are also (IH). Therefore, it is a sum of principal lanes. A sum of a sum of principal lanes is itself a sum of principal lanes. \square

Rewriting to a sum of principal lanes yields the same characteristic traces. This immediately follows, as an extra condition, from the proof of Lemma 6.20.

6.3 Normal Form and Completeness Theorem

To show that the set of axioms in Table 3 is complete we must show that any process part of a p -trace equivalence class can be rewritten to any other process in this class, using only the axioms from the set. This is achieved by picking one process from the class and showing that all other processes can be rewritten to it. We say that this process is in *normal form*.

In CCS, a normal form can be achieved by removing the non-deterministic behaviour from a process. We wonder if the same holds true for probabilistic processes. This is not the case. We can give a simple counter-example.

Example 6.21. Let $x = a.1b.1\delta + a.\downarrow\frac{1}{2}b.1\delta$. We have that $pTr\llbracket x \rrbracket = \{(\varepsilon, 1), (a, 1), (ab, 1), (ab, \frac{1}{2})\}$. We have two traces with the same action sequence, which by definition of pTr and operation semantics can only be the result of alternative composition. This means we can not rewrite x to a process $a.u$, for any process expression $u \in P(1)$.

We come up with a more elaborated normal form. In the previous sections, we introduced streets and lanes and presented a few properties. They form the building blocks to our normal form. We show that any probabilistic process can be rewritten to a normal form using the axioms. Then we show that a normal form is unique. This is the foundation of our completeness proof.

Definition 6.22. Let $x \in N$ be a probabilistic process. Then x is in normal form iff

- it is a sum of lanes, where each lane's characteristic trace corresponds to exactly one trace in $pTr\llbracket x \rrbracket$,
- each trace in $pTr\llbracket x \rrbracket$ corresponds to the characteristic trace of exactly one lane, where the lanes are all principal, and
- the lanes occur in the same arbitrary total order.

Theorem 6.23. Every probabilistic process can be rewritten to normal form, using the axioms from Table 3.

Proof: Let $x \in N$ be a process. We can rewrite x to a sum of lanes $s = \sum_{t \in S} t$ (Theorem 6.13). Note that the characteristic trace of each lane in this sum corresponds to exactly one trace in $pTr\llbracket s \rrbracket$ (Lemma 6.15). We prove that we can rewrite s to a sum of *principal* lanes u , so that each

trace in $pTr\llbracket u \rrbracket$ corresponds to a characteristic trace of exactly one lane. We witness two cases.

- Suppose that there is a trace $(\sigma, \rho) \in pTr\llbracket s \rrbracket$ for which there is no lane $t_1 \in S$ in the sum of s , such that $(\sigma, \rho) = \mathcal{C}(t_1)$. There must be a lane $t_2 \in S$ such that $(\sigma, \rho) \in pTr\llbracket t_2 \rrbracket$ (pigeonhole principle). We can rewrite t_2 to a sum on lanes that contains t_1 (Lemma 6.17).
- Suppose that there are two lanes $t_1, t_2 \in S$ in the sum of s , such that $\mathcal{C}(t_1) = \mathcal{C}(t_2)$ and we can rewrite s so that one of them is omitted (A1, A2, A3).

If we expand all lanes to sums as described, we get a larger sum, namely a sum containing *at least* one lane for each trace. We can rewrite this to a sum of *principal* lanes (Lemma 6.20), which we can rewrite to a sum of principal lanes containing *exactly* one lane for each trace (by the second bullet above). Let this lane be u .

The sum of lanes u satisfies the first four rules of our normal form. We show that we can rewrite u to a process z that also satisfies the last. Let $\preceq: \mathcal{T} \times \mathcal{T}$ be a particular total order. We can order the lanes in $u = \sum_{i=1}^n y_i$ such that for $1 \leq i \leq j \leq n$ it holds that $\mathcal{C}(y_i) \preceq \mathcal{C}(y_j)$ using axioms A1 and A2. Let the process that is the result of this ordering be z . Since the lanes themselves have not changed in z , the first four rules are still met. It also satisfies the last one and therefore it is in normal form. This proves we can rewrite any process to normal form under p -trace equivalence. \square

Lemma 6.24. Let $x, y \in N$ be two processes in the normal form such that $x =_{pTr} y$. Then $x = y$.

Proof: We know that by definition for each trace $(\sigma, \rho) \in pTr\llbracket x \rrbracket = pTr\llbracket y \rrbracket$ there is exactly one corresponding lane t_x in x such that $\mathcal{C}(t_x) = (\sigma, \rho)$. Similarly, there is one t_y such that $\mathcal{C}(t_y) = (\sigma, \rho)$. Since the lanes are all principal this means that $t_x = t_y$ (Lemma 6.19) and by extension that x and y contain exactly the same lanes. Also, they occur by definition in the same order, which proves that $x = y$. \square

Theorem 6.25. The axioms in Table 3 are complete.

Proof: Let $x, y \in N$ be two arbitrary p -trace equivalent processes. We have that $x =_{pTr} y$. We must show that we can rewrite x to y , and the other way around. We show the former since the proof for the converse

is symmetrical. We can rewrite x to normal form z (Lemma 6.23). Since $x =_{pTr} y$ we have that z is also the normal form of y (Lemma 6.24). Note that we can rewrite z to y (Lemma 6.23). We can rewrite x to y and vice versa. Therefore, the axioms are complete. \square

7 Conclusion

We have presented an axiomatisation for the probabilistic trace equivalence, characterised as weighted traces, for the alternating model of Hansson. We proved this axiomatisation is complete. We have forgone the question whether this equivalence is the right one. Also, the use of a deterministic, memoryless scheduler is debatable as it may cause the equivalence to become over-discriminating (Remark 4.11). Other notions, such as distributed schedulers [7, 6, 9], restricted schedulers [8] or coherent resolutions [3], address this issue. The use of less discriminating schedulers may give rise to a more general version of Axiom PA3. This has not been explored. The addition of the parallel operator and recursion has also been left unexplored. The latter would require a different strategy, as the employed normal form will no longer be adequate. Future work should explore these options, as well as opt for the more favourable simple Segala model.

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