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Maximum cardinality popular matchings in the stable marriage problem

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Abstract

Popular matching and was extensively studied in recent years as an alternative to stable matchings. Both type of matchings are defined in the framework of Stable Marriage (SM) problem: in a given bipartite graph $G = (A, B; E)$ each vertex u has a strict order of preference on its neighborhood. A matching M is popular, if for every matching M' of G, the number of vertices that prefer M' to M is at most the number of vertices which prefer M to M'. In this paper we prove that every maximum cardinality popular matching saturates the same set of vertices. This property is similar to that of stable matchings: any such matching saturates the same set of vertices.

Keywords: popular matching, stable matching, stable marriage problem.

1 Introduction

Our instance will be that of the Stable Marriage (SM) problem with strict preferences and incomplete lists: a bipartite graph $G = (A, B; E)$ (usually A is a set of men and B a set of women), every vertex has to be matched to (at most) one of its neighbors and ranks all its neighbors in a strict order of preference; these lists of preference are incomplete: G is not necessarily a complete bipartite graph.

A matching M of G is said to be *stable*[1] if no pair (man, woman) prefer each other to their respective matchings in M. This problem dates back to the famous paper of Gale and Shapley [2]; they give a polynomial time algorithm (with men proposing and women disposing) for finding a stable matching in the case of complete lists of preferences. This algorithm was extended for incomplete strict lists by Gusfield and Irving in [4]. The article of Gale and Shapley is the source of a whole long list of adjacent/generalized problems and algorithms, but the notion of weak stability has some draw-backs including that of being too restrictive, in the following sense: any stable matching saturates the same set of vertices (Gale and Sotomayor in [3]).

It was also was proved (in [3]) that the Gale-Shapley algorithm offers a stable matching in which every man has the best mate that can have in any stable matching. Perhaps this kind of optimality (Pareto efficiency) makes the stability so confining; as a consequence another, more attractive, notion of optimality has emerged: popularity. Popular matchings were introduced by Gärdenfors in [5] as a global form of stability, a matching M is *popular* if there is no matching for which more vertices are better-off than in M. He proved that any stable matching is a popular one showing in this way that popularity is a more relaxed notion than classic stability (see also [1]). It is worth noting that a stable matching is a minimum cardinality popular matching ([6]).

Huang and Kavitha in [6], proved that a maximum cardinality popular matching always exists for an instance of SM problem and give a polynomial time complexity algorithm for finding such a matching; furthermore, in [7], Kavitha improved this algorithm to a linear (in $|E|$) one. These results are important as the binary relation of popularity is not even transitive.

In this paper we prove a property similar to that of stable matchings, who always saturate the same set of vertices, but for maximum cardinality popular matchings.

Theorem 1.1. *Every maximum cardinality popular matching saturates the same set of vertices.*

We will prove this theorem using mainly Kavitha's algorithm (see [7]) and some results from [6] and [1]. The second section enumerates some definitions, notations and useful results, and the third section is devoted to the proof of our main result together with some final remarks.

2 Popular matchings

Let $G = (A, B; E)$ a bipartite graph, where A is the set of men and B is the set of women. Each man $a \in A$ and each woman $b \in B$ has a strict list of preference on its neighbors. If a person x *prefers* y_1 to y_2 we write $y_1 >_x y_2$; we suppose that x prefers any of its neighbors to being single, therefore $x <_x y$, for each $y \in N_G(x)$. Hence, for each $x \in A \cup B$ we have a total order $>_x$ on the set $N_G(x) \cup \{x\}$.

Let $\mathcal{M}_G$ the family of matchings in G. For a given matching $M \in \mathcal{M}_G$ and $x \in A \cup B$ we use the following notations: $M(x) = y$ if exists an edge $xy \in M$, and $M(x) = x$ if such an edge does not

[1] We use the weak stability condition.

exist. The set of M-saturated vertices is $S(M) = \{x : \exists xy \in M\}$, its complementary is the set of M-unsaturated or M-exposed vertices: $E(M) = (A \cup B) \setminus S(M)$.

If $A' \subseteq A, B' \subseteq B$, we denote by $\mathcal{GS}(A', B')$ the matching returned by the Gale-Shapley algorithm, with men (from A') proposing, on the subgraph $[A' \cup B']_G$. For the sake of completeness we remind here this algorithm.

```
M ← ∅;
for( u ∈ A' ∪ B')
   M(u) ← u; // every people is initially free
while (∃u, M(u) ≠ u and u is not terminated) {
   w ← pop(N_G(u)); // m proposes to w
   if (M(w) == ∅) {// w accepts m
      M ← M ∪ {mw};
      M(w) ← m; M(m) ← w;
      }
   elseif (M(w) <_w m) {// w dispose M(w) and accepts m
      M ← (M \ {M(w)w}) ∪ {mw};
      M(M(w)) ← ∅; M(m) ← w; M(w) ← m;
      }
   if (N_G(u) == ∅)
      u terminates;
   }
return M;
```

Let x a vertex and y, y' two other vertices, we define

$$vote_x(y, y') = \begin{cases} -1, & y <_x y' \\ 1, & y >_x y' \\ 0, & \text{otherwise} \end{cases} .$$

Let $M, M' \in \mathcal{M}_G$, for a vertex $x \in V(G)$ we define $vote_x(M, M') = vote_x(M(x), M'(x))$. For a given set of vertices $W \subseteq V(G)$, we use the notation $\mathcal{P}_W(M, M') = \{x \in W : vote_x(M, M') = 1\}$ for vertices in W which prefer M to M'. The set of vertices which prefer M to M' is $\mathcal{P}_{V(G)}(M, M') = \{x \in V(G) : vote_x(M, M') = 1\}$.

We also use the notations $\Phi_W(M, M') = |\mathcal{P}_W(M, M')| - |\mathcal{P}_W(M', M)|$, and $\Phi(M, M') = |\mathcal{P}_{V(G)}(M, M')| - |\mathcal{P}_{V(G)}(M', M)|$.

We say that M is *more popular* than M', and write $M \succeq M'$, if $\Phi(M, M') \geqslant 0$.

Definition 2.1. *A matching M is popular if, for any matching $M' \in \mathcal{M}_G$, $M \succeq M'$.*

Kavitha's algorithm for a largest popular matching (see [7]) is the following.

```
A_0 ← A; B_0 ← B; A_1, B_1 ← ∅; // set of vertices;
M ← ∅; // the mathching;
while(true) {
   M ← M ∪ GS(A_1, B);
   B_1 ← B_1 ∪ (S(M) ∩ B);
   B_0 ← B \ B_1;
   M ← M ∪ GS(A_0, B_0);
   if(A_0 ⊆ S(M))
      return M;
   A_1 ← A_1 ∪ (E(M) ∩ A_0);
   A_0 ← A \ A_1;
}
return M;
```

We label each edge $e = uv \in E(G) \setminus M$ with a pair $(\alpha_e, \beta_e) = (vote_u(v, M(u)), vote_v(u, M(v)) \in \{-1, 0, 1\}^2$.

Throughout the rest of the paper M will be the maximum cardinal popular matching returned by Kavitha's algorithm. Let G_M be the subgraph of G containing all non $(-1, -1)$ labeled edges. The following result which gathers different lemmata from [7] will be useful for our result.

Theorem 2.1. *([7]) If M is the matching returned by the above algorithm then*

(i) $M \subseteq (A_0 \times B_0) \cup (A_1 \times B_1)$.

(ii) $E(M) \subseteq A_1 \cup B_0$.

(iii) Every $(1,1)$ *labeled edge belongs to* $A_0 \times B_1$.

(iv) Every edge from $A_1 \times B_0$ *is labeled* $(-1,-1)$.

(v) If an alternating path in G_M, P, starts with a M-exposed vertex, then P does not contain $(1,1)$ *labeled edges.*

(vi) If an alternating path in G_M, P, starts with a M-saturated vertex, then P contains at most one $(1,1)$ *labeled edge.*

The following lemma has its importance for our result and it is proved in other form in [6], but for the sake of completeness we give here another short proof using Theorem 2.1.

Lemma 2.1. *There are no augmenting paths with respect to M in G_M.*

Proof. Suppose on the contrary that there exists an augmenting path $P = x_0, x_1, \dots x_{2p+1}$ with respect to M in G_M. Let $L = A_1 \cup B_0$ and $R = A_0 \cup B_1$. Obviously $x_0, x_{2p+1} \in L$; but $x_0 \in L$ implies $x_1 \in R$, as every edge from $A_1 \times B_0$ is $(-1,-1)$ labeled; now $x_1x_2 \in M$ implies $x_2 \in L$, and so on. In the end we get $x_{2p+1} \in R$ which is a contradiction.

□

Let $M_1, M_2 \in \mathcal{M}_G$ and consider the subgraph of G, $M_1 \oplus M_2 = (A \cup B, M_1 \Delta M_2)$. It is easy to see that the following lemma is true.

Lemma 2.2. *([1]) A matching M_1 is popular if and only if, for any other matching M_2, $|\mathcal{P}_{V(C)}(M_1, M_2)| \geqslant |\mathcal{P}_{V(C)}(M_2, M_1)|$, for each component, C, of $M_1 \oplus M_2$.*

3 Main result's proof and conclusions

Proof. Obviously $|\mathcal{P}(M, M')| = |\mathcal{P}(M', M)|$ and $|M| = |M'|$. We delete from M' the set N' of $(-1,-1)$ labeled edges and get another matching M'' with $|M''| = |M'| - k \leqslant |M|$, where $|N'| = k$.

Let us consider the subgraphs of G, $M \oplus M' = (A \cup B, M \Delta M')$, and $M \oplus M'' = (A \cup B, M \Delta M'')$; the connected components of these two subgraphs can be: isolated vertices, alternating cycles, or alternating paths.

We will show that $M \oplus M'$ contains only isolated vertices and alternating cycles.

Fact 3.1. *$vote_x(M'(x), M(x)) = vote_x(M''(x), M(x))$, for every vertex $x \in V(G)$.*

If $uv \in M'$ is labeled $(-1,-1)$, then $vote_u(M'(u), M(u)) = -1 = vote_v(M'(v), M(v))$, but

$$vote_u(M''(u), M(u)) = vote_u(u, M(u)) = -1 = vote_v(v, M(v)) = vote_v(M''(v), M(v)).$$

For all other vertices the equality is obvious.

Fact 3.2. *For every component, C, of $M \oplus M'$ we have $|\mathcal{P}_{V(C)}(M, M')| = |\mathcal{P}_{V(C)}(M', M)|$, hence $\Phi_{V(C)}(M, M') = 0$.*

The proof of this simple fact comes from Lemma 2.2.

Fact 3.3. *Every component of $M \oplus M''$ is included in a component of $M \oplus M'$, and a component of $M \oplus M''$ cannot be an augmenting path with respect to M.*

This is true because $M'' = M' \setminus N'$ and by use of Lemma 2.1.

Fact 3.4. *If P, an even alternating path, is a component of $M \oplus M''$, then $|\Phi_{V(P)}(M, M'') = 1$.*

If P is an even M, M''-alternating path, one of its endpoint will be a M-exposed vertex, and the other, v, will be M''-exposed and M saturated. Hence, from Theorem 2.1 (v), P does not contain any $(1,1)$ labeled edges: every edge of $E(P) \setminus M$ will be $(1,-1)$ or $(-1,1)$ labeled. Furthermore,

$$|\mathcal{P}_{V(P)}(M'', M)| - |\mathcal{P}_{V(P)}(M, M'')| = \sum_{e \in E(P) \setminus M} (\alpha_e + \beta_e) + vote_v(M''(v), M(v)) = -1.$$

Fact 3.5. *If P, an odd alternating path, is a component of $M \oplus M''$, then $\Phi(M, M'') \geqslant 0$.*

From Theorem 2.1, (vi), such a path must have at most one $(1,1)$ labeled edge, e', and its end-points, u and v, are M-saturated vertices. All other edges are $(-1,1)$ or $(1,-1)$ labeled edges. Therefore

$$|\mathcal{P}_{V(P)}(M'', M)| - |\mathcal{P}_{V(P)}(M, M'')| =$$

$$\sum_{e \in E(P) \setminus M} (\alpha_e + \beta_e) + vote_u(M''(u), M(u)) + vote_v(M''(v), M(v)) =$$

$$= \alpha_{e'} + \beta_{e'} - 2 \leqslant 2 - 2 = 0.$$

Fact 3.6. *Let P be an alternating path, component of $M \oplus M'$, and $(P_i)_{1 \leqslant i \leqslant k_P}$ all the components of $M \oplus M''$ included in P. Then*

$$\Phi_{V(P)}(M, M') = \sum_{i=1}^{k_p} \Phi_{V(P_i)}(M, M'').$$

If $P = P_1$, the conclusion is obviously true. Suppose now that $N' \cap E(P) \neq \varnothing$. Then, there is an order $\{i_1, i_2, \ldots, i_{k_P}\}$ and $N' \cap E(P) = \{u_1v_1, u_2v_2, \ldots, u_{k_P-1}v_{k_P-1}\}$, such that $u_l \in V(P_{i_l}), v_l \in V(P_{i_{l+1}})$, for every $1 \leqslant l \leqslant k_P - 1$. Therefore the votes from nodes incident with edges in N' are all numbered in $\sum_{i=1}^{k_p} \Phi_{V(P_i)}(M, M'')$.

Fact 3.7. *Let P be an alternating path, component of $M \oplus M'$, and $(P_i)_{1 \leqslant i \leqslant k_P}$ all the components of $M \oplus M''$ included in P. Then all the paths P_i must be augmenting with respect to M''.*

Using Facts 3.3 - 3.5, if a P_i is an even alternating path, then, from Fact 3.6, $\Phi_{V(P)}(M, M') \geqslant 1$, which is in contradiction with Fact 3.2. Therefore, all P_i's are augmenting with respect to M''.

Fact 3.8. *If P is an alternating path, component of $M \oplus M'$, then $|E(P) \cap M'| = |E(P) \cap M| - 1$.*

Let P be an alternating path, component of $M \oplus M'$, and $(P_i)_{1 \leqslant i \leqslant k_P}$ all the components of $M \oplus M''$ included in P. By Fact 3.7, all the paths P_i must be augmenting with respect to M''; these paths are linked by $(k_P - 1)$ edges from N'. Therefore

$$|E(P) \cap M'| = (k_P - 1) + \sum_{i=1}^{k_P} |E(P_i) \cap M'| =$$

$$= (k_P - 1) + \sum_{i=1}^{k_P} (|E(P_i) \cap M| - 1) = |E(P) \cap M| - 1.$$

Fact 3.9. *$M \oplus M'$ cannot have connected components which are alternating paths.*

If a component C in $M \oplus M'$ is not and alternating path, then $|E(C) \cap M'| = |E(C) \cap M|$, as C must be an isolated vertex or an alternating cycle. Let $\mathcal{C}$ the family of connected components of $M \oplus M'$, we have

$$\sum_{C \in \mathcal{C}} |E(C) \cap M'| = |M' \setminus M|, \sum_{C \in \mathcal{C}} |E(C) \cap M| = |M \setminus M'|,$$

if one of the components in $M \oplus M'$ is an alternating path, then $|M' \setminus M| < |M \setminus M'|$, which implies $|M| > M'|$.

Fact 3.9 tells us that $M \oplus M'$ contains only isolated vertices and alternating cycles as connected components, hence $S(M) = S(M')$.

□

Example 3.1. *There are three men and three women with preference lists as follows:*

$$\begin{array}{ll} m_1 : \; w_1 \, w_2 & w_1 : \; m_2 \, m_1 \\ m_2 : \; w_1 \, w_3 \, w_2 & w_2 : \; m_2 \, m_1 \\ m_3 : \; w_3 & w_3 : \; m_2 \, m_3 \end{array}$$

This instance of SM has four maximal matchings

$$M_1 = \{m_1w_1, m_2w_2, m_3w_3\}, \quad M_2 = \{m_1w_1, m_2w_3\}$$
$$M_3 = \{m_1w_2, m_2w_1, m_3w_3\}, \quad M_4 = \{m_1w_2, m_2w_3\}$$

M_1 *is not a stable matching (as* $m_2 >_{w_3} M_1(w_3) = m_3$ *and* $w_3 >_{m_2} M_1(m_2) = w_2$*), and* M_3 *is a stable one;* M_1 *and* M_3 *are popular:*

$$\Phi(M_1, M_2) = 2, \Phi(M_1, M_3) = 0, \Phi(M_1, M_4) = 0.$$

Obviously, both these popular matchings saturates the same set, namely $V(G)$*.* M_1 *being a stable matching can be returned by Kavitha's algorithm, but is not men-optimal among all maximum cardinality popular matchings:* $M_1(m_2) = w_2 <_{m_2} M_3(m_2) = w_1$.

The above example shows that the matchings produced by this algorithm are not Pareto optimal among maximum cardinality popular matchings as in the case of Gale-Shapley algorithm for the class of stable matching.

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