The Theory of Finitely Supported Structures and Choice Forms

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Abstract

The theory of finitely supported algebraic structures provides a first step in computing infinite algebraic structures that are finitely supported modulo certain atomic permutation actions. The motivation for developing such a theory comes from both mathematics (by modelling infinite algebraic structures, hierarchically defined by involving some basic elements called atoms, in a finitary manner, by analyzing their finite supports) and computer science (where finitely supported sets are used in various areas such as semantics foundation, automata theory, domain theory, proof theory and software verification). The results presented in this paper include the meta-theoretical presentation of finitely supported structures, the study of the consistency of choice principles (and of results requiring choice principles) within this framework, and the presentation of several connections with other topics.

Keywords: finitely supported set, choice principle, logical notion.

1 Introduction and Related Topics

There does not exist a formal description of 'infinite' in those sciences which are focused on quantitative aspects. Questions such as 'What represents the infinite?', 'How could the infinite be modelled?', or 'Does the infinite really exist or is it just a convention?' naturally appear. In order to provide some appropriate answers, we present 'Finitely Supported Mathematics (FSM)', (which is a general name for 'the theory of finitely supported

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algebraic structures'). FSM is developed by employing the general principle of finite support, claiming that any infinite structure, hierarchically defined by involving some basic elements called atoms, must have a finite support, that is, FSM is obtained from the classical algebra by replacing 'non-atomic structure' with 'finitely supported atomic structure'. Informally, in Finitely Supported Mathematics we can model infinite structures (hierarchically constructed over atoms) by using only a finite number of characteristics. More precisely, in FSM we admit the existence of infinite structures, but for an infinite structure we ascertain that only a finite family of its elements (i.e. its 'finite support') is "really important" in order to characterize the related structure, while the other elements are somehow "similar". The following topics are related to FSM.

The Fraenkel-Mostowski (FM) permutation models of Zermelo Fraenkel set theory with atoms (ZFA) were developed in 1930s by Fraenkel, Lindenbaum and Mostowski [8, 17] in order to prove the independence of axiom of choice (**AC**) from the other axioms of ZFA set theory, but they have been recently rediscovered by Gabbay and Pitts in order to describe syntax involving binding operations. Several models of ZFA set theory are well known. We mention permutative Fraenkel basic and second models, the permutative Mostowski ordered model [14], and the von-Neumann cumulative hierarchy $\nu(A)$ of sets involving atoms from the set of atoms A described as follows:

- $\nu_0(A) = \emptyset$;

- $\nu_{\alpha+1}(A) = A + \wp(\nu_\alpha(A))$ for every non-limit ordinal α;

- $\nu_\lambda(A) = \underset{\alpha<\lambda}{\cup} \nu_\alpha(A)$ (λ a limit ordinal);

- $\nu(A) = \underset{\alpha}{\cup}\nu_\alpha(A)$,

where $+$ is the disjoint union of sets (used to emphasize the difference between sets and atoms), and $\wp(X)$ represents the powerset of X; $\nu(A)$ will be in depth analyzed in Section 4.

FM axiomatic set theory was presented in [9]. It is inspired by FM permutation models of ZFA set theory. However, FM set theory, ZFA set theory and Zermelo-Fraenkel (ZF) set theory are independent axiomatic set theories. All of these theories are described by axioms, and all of them have models. For example, the Cumulative Hierarchy Fraenkel-Mostowski universe $FM(A)$ presented in [9] is a model of FM set theory, while detailed

lists of Cohen models of ZF and Fraenkel-Mostowski permutation models of ZFA can be found in [13]. The axioms of FM set theory are precisely the ZFA axioms over an infinite set of atoms, together with the special axiom of finite support which claims that for each element x in an arbitrary set we can find a finite set supporting x according to the hierarchically constructed group action of the group of all permutation of atoms. The original purpose of axiomatic FM set theory was to provide a mathematical model for variables in a certain syntax. Since they have no internal structure, atoms can be used to represent names. The finite support axiom is motivated by the fact that syntax can only involve finitely many names. The construction of the universe of all FM-sets [9] is inspired by the construction of the universe of all admissible sets over an arbitrary collection of atoms [5]. The FM-sets represent a generalization of hereditary finite sets (which are particular admissible sets used to describe 'Gandy machines' [10]); actually, any FM-set is an hereditary finitely supported set.

Nominal sets represent a ZF alternative to the non-standard FM set theory. More exactly, nominal sets can be defined both in the ZF framework [19] and in the FM framework [9]. In ZF, a nominal set is defined as a usual ZF set endowed with a particular group action of the group of permutations over a certain fixed set A (also called the set of atoms by analogy with the FM framework), that satisfies a finite support requirement. There exists also an alternative definition for nominal sets in the FM framework (when the fixed ZF set A is replaced by the set of atoms in FM). They can be defined as sets constructed according to the FM axioms with the additional property of being empty supported (invariant under all permutations of atoms). These two ways of defining nominal sets finally lead to similar properties. Moreover, we can say that any set defined according to the FM axioms (any FM-set) can be seen as a subset of the nominal set $FM(A)$. However, an FM-set is itself a nominal set only if it has an empty support. Intuitively, in a lambda-calculus interpretation, we can think of the elements of a nominal set as having a finite set of 'free names'. The action of a permutation on such an element actually represents the renaming of the 'bound names'. In computer science, nominal sets offer an elegant formalism for describing λ-terms modulo α-conversion [9, 19]. They can also be used in domain theory [20, 19], in the theory of abstract interpretation [2], in topology [18], and programming [20]. Since the principles of structural recursion and induction were proved to be consistent in the framework of nominal sets [19], we ascertain that the theory of nominal sets provides a

right balance between an informal reasoning and a rigorous formalism.

Similarly to Klein's program for the classifications of geometries [15], Tarski defined the logical notions as those invariant under all possible one-to-one transformations of the universe of discourse onto itself [21]. In [3] the authors proved that those nominal sets defined in the FM cumulative universe $FM(A)$ are logical in Tarski's view. This is because in the FM framework it can be proved (see Lemma 1 from [3]) that every one-to-one transformation of A onto itself must be a finite permutation of A (i.e. a permutation that leaves unchanged all but finitely many elements of A), and so the effect of a bijective self-transformation of the universe of discourse on an element x belonging to a nominal (empty-supported) set $(X, \cdot)$ from $FM(A)$ coincides with the effect of a finite permutation of atoms on x under the group action $\cdot$. Such an element x is left invariant under the effect of any bijection of atoms if and only if it is equivariant (empty supported) as an element of X. By considering $X = FM(A)$, those equivariant elements of X are precisely the nominal FM sets.

Generalized nominal sets represent a generalization of the theory of nominal sets over a fixed set A of atoms to a new theory of nominal sets over arbitrary (unfixed) sets of data values (that may have an internal structure)[6]. The notion of 'S_A-set' (Definition 2) is replaced by the notion of 'set endowed with an action of a subgroup of the symmetric group of $\mathbb{D}$' for an arbitrary set of data values $\mathbb{D}$, and the notion of 'finite set' is replaced by the notion of 'set with a finite number of orbits with respect to the previous group action (orbit-finite set)'. This approach is useful for studying automata on data words, languages over infinite alphabets, or Turing machines that operate over infinite alphabets.

2   Principles of Defining FSM

In order to define Finitely Supported Mathematics - FSM, we use nominal sets (without the requirement that the set of atoms is countable) which from now on will be called invariant sets motivated by Tarski's approach regarding logicality. In FSM we actually study the finitely supported subsets of invariant sets together with finitely supported relations (order relations, functions, algebraic laws etc), and so FSM becomes a theory of atomic algebraic structures constructed/defined according to the finite support requirement.

Our goal is not to rebrand the nominal framework (with significant

applications in computer science), but we work on foundations of mathematics by proving a collection of new set theoretical results regarding finitely supported sets, and for this purpose we consider that 'invariant' and 'FSM' are more appropriate names (in the sense of Tarski definition of logical notions as those invariant under permutations).

Adjoin to ZF a infinite (possibly uncountable) set A formed by elements whose internal structure is irrelevant. In this sense we will refer to A as being a fixed ('special') infinite ZF set which will be called 'the set of atoms' by analogy with ZFA approach. The set theoretical (hierarchical) constructions over atoms will be called 'atomic' (elements, sets, structures, etc) and will be emphasized separately from the classical ZF constructions (which are non atomic, i.e. they are hierarchically constructed from the empty set). An invariant set $(X, \cdot)$ is actually a classical ZF set X equipped with an action $\cdot$ on X of the group of permutations of A, having the additional property that any element $x \in X$ is finitely supported. In a pair $(X, \cdot)$ formed by a ZF set X and a group action $\cdot$ on X of the group of all permutations of A, an arbitrary element $x \in X$ is finitely supported if there exists a finite family $S \subseteq A$ such that any permutation of A that fixes S pointwise also leaves x invariant under the group action $\cdot$. An empty supported element $x \in X$ is called equivariant. A relation (or, particularly, a function) between two invariant sets is finitely supported/equivariant if it is finitely supported/equivariant as a subset of the Cartesian product of those two invariant sets. Whenever an element $a \in A$ appears in (the construction of) an invariant set $(X, \cdot)$ (particularly if $X = A$), the effect of a permutation π of A on a under $\cdot$ is $\pi(a)$. Ordinary ZF sets defined/constructed without involving atoms (without involving elements of A) are trivial invariant sets. Generally, an algebraic structure is invariant (or finitely supported) if it can be represented as an invariant set (or as a finitely supported subset of an invariant set) endowed with an equivariant algebraic law (or with an algebraic law which is finitely supported as a subset of an invariant set). Detailed formal definitions can be found in Section 4. Concretely, FSM represents a reformulation of the ZF algebra obtained by replacing '(infinite) set' with 'invariant/finitely supported set'. All the structures defined in FSM (either atomic or non-atomic ones) must be finitely supported according to canonical hierarchically defined permutation actions (the non-atomic ones are trivial, while the atomic ones are equipped with canonical actions described in Example 1). FSM is consistent with the ZF axioms (by considering A as a fixed ZF set), but it can be adequately reformulated according to ZFA axioms (if the fixed ZF set A considered in its construction is replaced by the set of atoms in ZFA).

The main idea of reformulating a classical ZF result into FSM is to analyze if there exists a valid result obtained by replacing "structure" with "invariant/finitely supported structure (under canonical permutation action)" in the ZF result. Since every ordinary (non-atomic) ZF set is a particular invariant set equipped with a trivial permutation action, the general properties of (trivial) invariant sets lead to valid properties of ordinary ZF sets. Thus, FSM properties obtained for trivial actions lead to valid ZF properties. However, not every ZF result can be directly rephrased in terms of finitely supported objects according to canonical permutation actions (hierarchically defined an in Example 1). This is because, given an invariant set X, there could exist some subsets of X which fail to be finitely supported. A related example is represented by the subsets of the set A of atoms which are at the same time infinite and coinfinite. Intuitively, FSM contains the family of 'non-atomic' (ordinary) ZF sets (which are proved to be trivial FSM sets) and the family of 'atomic' sets (hierarchically constructed from the fixed ZF set A); the question is if a classical ZF result (obtained in ZF for non-atomic sets) can be adequately reformulated by replacing 'set' with 'finitely supported set' in order to remain valid also for atomic sets with finite support. Examples of valid ZF results that cannot be reformulated into FSM are presented in this paper. We particularly mention the independence of the axiom of choice and its weaker forms from ZF axioms, results regarding cardinalities of Dedekind-finiteness, as well as the Stone duality (see [18]).

In order to reformulate a general ZF result into FSM, the proof of the related FSM result should not break the principle that any construction has to be finitely supported, which means that the related proof should be internally consistent in FSM and not retrieved from ZF.

FSM is not a model of set theory, but an independent set theory whose axioms are the ZF axioms where we adjoin a infinite ZF set A (formed by basic elements, i.e. elements whose internal structure is irrelevant) and impose the additional axiom of finite supportness for all the constructions involving elements of A. Alternatively, A can be considered as the set of atoms in ZFA (obtained by weakening ZF axiom of exensionality), and so FSM corresponds to FM axiomatic set theory. This means that non-finitely supported structures are not allowed in this theory. They simply can appear from nowhere, even in an intermediate step of a proof. This is why no ZF (or ZFA) result can be used in FSM before reproving it in terms of finitely supported objects. Informally, when working in FSM, we cannot use a result outside FSM to prove something in FSM.

For studying the validity of an FSM result, one must prove that all the structures/constructions involved in the statement or in the proof of the related result are finitely supported. We present three general methods of proving that a certain structure is finitely supported.

- The first method is a "hierarchically constructive" one for supports.

- The second method is represented by a general equivariance/finite support principle which is defined by using the higher-order logic [19].

- The third method is given by a refinement of the equivariance/finite support principle allowing to prove boundedness properties for supports.

The hierarchically constructive method for supports represents an hierarchical (step-by-step) construction of the support of a certain structure by employing the supports of the sub-structures of a related structure. It means to anticipate a possible (intuitive) candidate for the support of an object (this candidate could be, for example, the empty set, the union of the supports of the sub-structures of the related object when this union is finite etc), and then to prove that this candidate is indeed a support by using the classical properties of finitely supported sets.

Equivariance/Finite support principle states that any function or relation that is defined from equivariant /finitely supported functions and subsets using classical higher-order logic is itself equivariant/ finitely supported, with requirement that we restrict any quantification over functions or subsets to range over ones that are equivariant/finitely supported. However, this method has some limitations because, as it is carried out in [19], it does not emphasize directly relationship properties between the supports of the structures involved in a higher-order construction. This item was solved by describing the next method which is a refinement of the Equivariance/Finite Support principle.

The S-finite support principle states that for any finite set $S \subseteq A$, anything that is definable (in higher order logic) from S-supported structures using S-supported constructions is S-supported. However, the formal application of this method generally overlaps on the constructive method for supports.

3 Limits of Transferability of Zermelo-Fraenkel Results in FSM

Although Finitely Supported Mathematics has historical origins in the construction of permutative models for ZFA set theory, it is independent from them (it is actually related to the theory of nominal sets). The formal connections between Finitely Supported Mathematics and the axiomatic theory of FM sets will be made clear in Section 4. It is worth noting that Finitely Supported Mathematics can be adequately presented in ZFA, if the fixed ZF set A used in its construction is replaced by the set of atoms in ZFA; this is because we did not require an internal structure for the elements of A. Since we can identify the FM sets with (hereditary) finitely supported subsets of certain invariant sets, the results in Finitely Supported Mathematics can be adequately rephrased also in the world of FM sets. Summarizing, FSM is a theory of algebraic structures obtain by replacing the term 'ZF structure' by 'finitely supported structure (under the canonical group action of the group of all permutations of atoms)', which can be as well reformulated over ZFA. However, there are many limitations when reformulating the results from ZF (or ZFA) into FSM. We emphasize various examples of non-transferable results between related frameworks such as ZF, ZFA and FSM.

Jech-Sochor's theorem (Theorem 6.1 in [14]) states that permutation models of ZFA can be embedded into symmetric models of ZF. As a consequence, a statement which holds in a given permutation model of ZFA and whose validity depend only on a certain fragment of that model, also holds in some well-founded model of ZF. Therefore, the proof of consistency in ZF for a certain existential statement reduces to the construction of a permutation model of ZFA. For example, the existence of sets that cannot be well ordered and of sets that cannot be totally ordered is consistent with ZF because such sets exist in (models of) ZFA. Similarly, by constructing adequate permutation models of ZFA and by involving Jech-Schor's embedding theorem, we can prove that the following statements are consistent with ZF: 'There exists a vector space which has no basis' and 'There exists a free group such that not every subgroup of it is free'. However, there does not exist a full equivalence between the consistency of results in ZF and the consistency of the appropriate results in ZFA.

The ZF relationship results does not necessarily remain valid when reformulate them in ZFA. There exist statements that are equivalent to

axiom of choice in ZF, but are weaker than axiom of choice in ZFA. For example, multiple choice principle (Definition 2.5 in [12]) and Kurepa's antichain principle (claiming that each partially ordered set has a maximal subset of pairwise incomparable elements) are such statements. According to Theorem 5.4 in [11], multiple choice principle and Kurepa's antichain principle are both equivalent to the axiom of choice in ZF. However, in Theorem 9.2 of [14] it is proved that multiple choice principle is valid in the Second Fraenkel Model (model N2 from [13]), while the axiom of choice fails in this model. Furthermore, Kurepa's maximal antichain principle is valid in the Basic Fraenkel Model (model N1 from [13]), while multiple choice principle fails in this model. This means that the following two statements (valid in ZF) 'Kurepa's principle implies axiom of choice' and 'Multiple choice principle implies axiom of choice' fail in ZFA.

The construction of FSM makes sense on both ZF and ZFA frameworks. However, the ZF and ZFA results are not necessary valid in FSM. This is because the family of sets in FSM (formed by invariant sets or finitely supported subsets of invariant sets) is not closed under subsets construction. Thus, for translating a ZF or ZFA result into FSM we must analyze if there exists a valid result obtained by replacing "object" with "finitely supported object" in the ZF result, or in the ZFA result respectively. Thus, for proving results in Finitely Supported Mathematics we cannot use related results from Zermelo-Fraenkel set theory without reformulating them with respect to the finite support requirement. We always need to verify if the proof of such a result employs only finitely supported constructions under canonical atomic actions.

The ZF results are not necessarily valid when we reformulate them in FSM. Various choice principles are proved to be independent from ZF axioms. For example, the ordering principle claiming that every set can be totally ordered is satisfied in Cohen's First Model (model M1 from [13]). However, this principle is not satisfied in Shelah's Second Model (model M38 from [13]), meaning that the ordering principle is independent from the ZF axioms. The axiom of dependent choice (Definition 2.11 in [12]) is valid in Shelah's Second Model but fails in Cohen's First Model, meaning that it also independent from the axioms of ZF set theory. The axiom of countable choice (Definition 2.5 in [12]) is valid in Shelah's Second Model but fails in Cohen's First Model, meaning that this principle is also independent from the axioms of ZF set theory. The prime ideal theorem (Definition 2.15 in [12]) is independent from the axioms of ZF set theory because it is valid

in Cohen's First Model, but it fails in Shelah's Second Model. Although the previously mentioned choice principles are independent from the ZF axioms, in Section 5 we prove that all of them are inconsistent with FSM.

Similarly, the ZFA results are not necessarily valid in FSM. Various choice principles are proved to be independent from ZFA axioms. For example, the ordering principle is satisfied in Mostowski Ordered Model (model N3 in [13]), but it fails in Fraenkel's Second Model, which means the ordering principle is independent from the axioms of ZFA [14]. The prime ideal theorem is satisfied in Mostowski Ordered Model (Theorem 7.16 in [11]), but it fails in Fraenkel's Second Model [14], which means the prime ideal theorem is independent from the axioms of ZFA. The countable choice principle is also independent from the axioms of ZFA because it is satisfied in Howard-Rubin's First Model (model N38 from [13]), but it fails in Fraenkel's Second Model [12]. Although the previously mentioned choice principles are independent from the ZFA axioms, they are all inconsistent with FSM, when FSM is obtained by replacing the fixed ZF set A with the set of atoms in ZFA (see Section 5).

4   Finitely Supported Sets: Formal Presentation

Adjoin to ZF a special infinite set A (formed by elements whose internal structure is irrelevant) that will be referred as fixed infinite, 'special' ZF set, and will be called 'the set of atoms' by analogy with ZFA approach. The following results will remain valid if the fixed ZF set A will be replaced precisely by the set of atoms (elements with no internal structure) form ZFA obtained by modifying/weakening the ZF axiom of extensionality and if 'ZF' will be replaced by 'ZFA' in their statement. The equivalence of various definitions of finiteness is a consequence of **AC**. To avoid any doubt, where the word "finite" appears in this paper, it means "it corresponds one-to-one and onto to a finite ordinal".

Definition 1   *i) A* transposition *is a function* $(a\,b) : A \to A$ *defined by* $(a\,b)(a) = b$, $(a\,b)(b) = a$ *and* $(a\,b)(n) = n$ *for* $n \neq a, b$.

   ii) A permutation *of A is a one-to-one and onto function on A which moves only finitely many elements.*

Let S_A be the group of all permutations (i.e. the set of those one-to-one transformations of A onto itself which can be expressed by composing

finitely many transpositions). It can be proved that an arbitrary one-to-one transformations of A onto itself is finitely supported if and only if it is a permutation of A in the sense of Definition 1 (see Lemma 1 from [3]); this is why we considered only (finite) permutations of A instead of arbitrary one-to-one transformations of A onto itself.

Definition 2   • *Let X be a ZF-set. An S_A-action on X is a function $\cdot : S_A \times X \to X$ having the properties that $Id \cdot x = x$ and $\pi \cdot (\pi' \cdot x) = (\pi \circ \pi') \cdot x$ for all $\pi, \pi' \in S_A$ and $x \in X$.*

• *An S_A-set is a pair $(X, \cdot)$ where X is a ZF-set, and $\cdot : S_A \times X \to X$ is an S_A-action on X. We simply use X whenever no confusion arises.*

• *Let $(X, \cdot)$ be an S_A-set. We say that $S \subset A$ supports x (or x is supported by S) whenever for each $\pi \in Fix(S)$ we have $\pi \cdot x = x$, where $Fix(S) = \{\pi \mid \pi(a) = a, \forall a \in S\}$. An empty supported element of X is called* equivariant.

• *Let $(X, \cdot)$ be an S_A-set. We say that X is an* invariant set *if for each $x \in X$ there exists a finite set $S_x \subset A$ which supports x. The intersection of all finite sets supporting x also supports x (i.e. it is the least finite set supporting x) and it is called* the support of x, *and we denote it by $supp(x)$.*

Proposition 1 ([1]) *Let $(X, \cdot)$ be an S_A-set and let $\pi \in S_A$ be an arbitrary permutation. Then for each $x \in X$ which is finitely supported we have that $\pi \cdot x$ is finitely supported and $supp(\pi \cdot x) = \pi(supp(x))$.*

Example 1 *The following items present how the canonical permutation action is constructed on certain sets. Their proofs are direct consequences of Definition 2 - see [1].*

1. *The set A of atoms is an S_A-set with the* canonical S_A-action $\cdot : S_A \times A \to A$ *defined by $\pi \cdot a := \pi(a)$ for all $\pi \in S_A$ and $a \in A$. $(A, \cdot)$ is an invariant set because for each $a \in A$ it follows that $supp(a) = \{a\}$.*

2. *The set S_A is an S_A-set with the S_A-action $\cdot : S_A \times S_A \to S_A$ defined by $\pi \cdot \sigma := \pi \circ \sigma \circ \pi^{-1}$ for all $\pi, \sigma \in S_A$. $(S_A, \cdot)$ is an invariant set because for each $\sigma \in S_A$ it follows that $supp(\sigma) = \{a \in A \mid \sigma(a) \neq a\}$.*

3. *Any ordinary (non-atomic) ZF-set X (such as $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{R}$ for example) is an S_A-set with the (single possbile) S_A-action $\cdot : S_A \times X \to X$ defined by $\pi \cdot x := x$ for all $\pi \in S_A$ and $x \in X$. Also X is an invariant set because for each $x \in X$ it follows that $supp(x) = \emptyset$.*

4. *If $(X, \cdot)$ is an S_A-set, then $\wp(X) = \{Y \mid Y \subseteq X\}$ is also an S_A-set with the S_A-action $\star : S_A \times \wp(X) \to \wp(X)$ defined by $\pi \star Y := \{\pi \cdot y \mid y \in Y\}$ for all $\pi \in S_A$, and all $Y \subseteq X$. Note that $\wp(X)$ is not necessarily an invariant set even if X is. For each S_A-set $(X, \cdot)$, we denote by $\wp_{fs}(X)$ the set formed from those subsets of X which are finitely supported as elements of $\wp(X)$ under the action $\star$. According to Proposition 1, $(\wp_{fs}(X), \star|_{\wp_{fs}(X)})$ is an invariant set, where $\star|_{\wp_{fs}(X)} : S_A \times \wp_{fs}(X) \to \wp_{fs}(X)$ is defined by $\pi \star|_{\wp_{fs}(X)} Y := \pi \star Y$ for all $\pi \in S_A$ and $Y \in \wp_{fs}(X)$.*

5. *If $(X, \cdot)$ is an S_A-set, then the finite power set of X, $\wp_{fin}(X) = \{Y \subseteq X \mid Y \, finite\}$ and the cofinite power set of X, $\wp_{cofin}(X) = \{Y \subseteq X \mid X \setminus Y \, finite\}$ are S_A-sets with the S_A-action $\star$ defined as at item 4. If X is an invariant set, then both $\wp_{fin}(X)$ and $\wp_{cofin}(X)$ are invariant sets. If $B \in \wp_{fin}(A)$, then $supp(B) = B$. If $C \in \wp_{cofin}(A)$, then $supp(C) = A \setminus C$. Furthermore, $\wp_{fs}(A) = \wp_{fin}(A) \cup \wp_{cofin}(A)$.*

6. *Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. The Cartesian product $X \times Y$ is also an S_A-set with the S_A-action $\star : S_A \times (X \times Y) \to (X \times Y)$ defined by $\pi \star (x, y) = (\pi \cdot x, \pi \diamond y)$ for all $\pi \in S_A$ and all $x \in X$, $y \in Y$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X \times Y, \star)$ is also an invariant set.*

7. *Let $(X, \cdot)$ and $(Y, \diamond)$ be S_A-sets. We define the disjoint union of X and Y by $X + Y = \{(0, x) \mid x \in X\} \cup \{(1, y) \mid y \in Y\}$. $X + Y$ is an S_A-set with the S_A-action $\star : S_A \times (X + Y) \to (X + Y)$ defined by $\pi \star z = (0, \pi \cdot x)$ if $z = (0, x)$ and $\pi \star z = (1, \pi \diamond y)$ if $z = (1, y)$. If $(X, \cdot)$ and $(Y, \diamond)$ are invariant sets, then $(X + Y, \star)$ is also an invariant set: each $z \in X + Y$ is either of the form $(0, x)$ and supported by the finite set supporting x in X, or of the form $(1, y)$ and supported by the finite set supporting y in Y.*

Definition 3 *Let $(X, \cdot)$ be an S_A-set. A subset Z of X is called* finitely supported *if and only if $Z \in \wp_{fs}(X)$. An equivariant subset of an invariant set is itself an invariant set.*

In FSM we will consider that A is the invariant set equipped with the canonical S_A-action defined on Example 1(1). Any invariant set should be equipped with the canonical action hierarchically constructed as in Example 1 (for power sets, Cartesian products, unions etc). Whenever an atom a appears in (the construction of) an invariant set $(X, \cdot)$, the effect of a permutation of atoms π on a under $\cdot$ will be $\pi(a)$. However, for ordinary ZF sets defined without involving any atoms, the trivial actions are the only-possible S_A-actions on these sets such that these sets are invariant. Intuitively, if we consider the set of all positive integers $\mathbb{N}$, we know that $\mathbb{N}$ is defined by starting with $\emptyset$, and then forming the sets $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, and so on. Since $\emptyset$ is not defined by employing atoms, it cannot be changed under the effect of a certain permutation of atoms.

Recall that a function $f : X \to Y$ is a particular relation on $X \times Y$.

Definition 4 *Let X and Y be invariant sets. A function $f : X \to Y$ is finitely supported if $f \in \wp_{fs}(X \times Y)$.*

Let $Y^X = \{f \subseteq X \times Y \mid f$ is a function from the underlying set of X to the underlying set of $Y\}$.

Proposition 2 ([1]) *Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets. Then Y^X is an S_A-set with the S_A-action $\star : S_A \times Y^X \to Y^X$ defined by $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. A function $f : X \to Y$ is finitely supported in the sense of Definition 4 if and only if it is finitely supported with respect to the permutation action $\star$.*

Proposition 3 ([1]) *Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets. Let $f \in Y^X$ and $\sigma \in S_A$ be arbitrary elements. Let $\star : S_A \times Y^X \to Y^X$ be the S_A-action on Y^X, defined by: $(\pi \star f)(x) = \pi \diamond (f(\pi^{-1} \cdot x))$ for all $\pi \in S_A$, $f \in Y^X$ and $x \in X$. Then $\sigma \star f = f$ if and only if for all $x \in X$ we have $f(\sigma \cdot x) = \sigma \diamond f(x)$.*

Definition 4 can be generalized in the following way:

Definition 5 *Let X and Y be invariant sets, and let Z be a finitely supported subset of X, T be a finitely supported subset of Y. A function $f : Z \to T$ is finitely supported if $f \in \wp_{fs}(X \times Y)$.*

Proposition 4 *Let $(X, \cdot)$ and $(Y, \diamond)$ be invariant sets, and let Z be a finitely supported subset of X and T be a finitely supported subset of Y. Let $f : Z \to T$ be a function. The function f is finitely supported in the sense of Definition 5 by a finite set S of atoms if and only if for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$, $\pi \diamond f(x) \in T$ and $f(\pi \cdot x) = \pi \diamond f(x)$.*

Proof: We assume that f is finitely supported in the sense of Definition 5 by S. Then $\pi \star f = f$ for all $\pi \in Fix(S)$, where $\star$ represents the S_A-action on $\wp(X \times Y)$ defined as in Example 1(4). Let $x \in Z$ and $\pi \in Fix(S)$ be arbitrary elements. Then there exists an unique $y \in T$ such that $(x, y) \in f$. Since $\pi \star f = f$, we have $(\pi \cdot x, \pi \diamond y) \in f \subseteq (Z \times T)$. Thus, $\pi \cdot x \in Z$, $\pi \diamond f(x) \in T$ and $f(\pi \cdot x) = \pi \diamond y = \pi \diamond f(x)$.

Conversely, we assume that there exists a finite set S of atoms such that for all $x \in Z$ and all $\pi \in Fix(S)$ we have $\pi \cdot x \in Z$, $\pi \diamond f(x) \in T$ and $f(\pi \cdot x) = \pi \diamond f(x)$. We claim that $\pi \star f = f$ for all $\pi \in Fix(S)$. Fix some $\pi \in Fix(S)$, and consider (x, y) an arbitrary element in f. We have $f(x) = y$, and so $(\pi \cdot x, \pi \diamond y) \in f$. However, $\pi \rhd (x, y) = (\pi \cdot x, \pi \diamond y) \in f$, where $\rhd$ represents the S_A-action on $X \times Y$ defined as in Example 1(6). This means $\pi \star f = f$. □

The previous results are valid if A is an arbitrary fixed infinite ZF set. Since we did not require a certain internal structure of the elements of A, these results can be reformulated in the ZFA framework by replacing the fixed ZF set A with the set of atoms in ZFA (also denoted by A). If A is a set of atoms in ZFA (elements with no internal structure), as in [9] we can slightly modify the usual von-Neumann hierarchy on ordinals and define a ZFA class (i.e. a 'large' set) $FM(A)$ equipped with an S_A-action, in which all the elements have the finite support property.

Given the set A of atoms in ZFA (described by modifying/weakening the ZF Axiom of Extensionality), we remind that a cumulative hierarchy of sets involving atoms from A can be defined recursively by considering $\nu_0(A) = \emptyset$; $\nu_{\alpha+1}(A) = A + \wp(\nu_\alpha(A))$ for α a non-limit ordinal, and $\nu_\lambda(A) = \underset{\alpha < \lambda}{\cup} \nu_\alpha(A)$ for λ a limit ordinal. Let $\nu(A)$ be the union of all $\nu_\alpha(A)$. It is clear that $\nu(A)$ is a model of ZFA set theory, and obviously, the standard ZF sets (hierarchically constructed over $\emptyset$) belong to $\nu(A)$. Using the names atm and set for the functions $x \mapsto (0, x)$ and $x \mapsto (1, x)$ (the notations are those used in Example 1(7)) we have that every element x of $\nu(A)$ is either of the form $atm(a)$ with $a \in A$, or of the form $set(X)$ where X is a set formed at an earlier ordinal stage than x. We call ZFA sets the elements of the form $set(X)$, and atoms the elements of the form $atm(a)$.

On $\nu(A)$ we can recursively define an S_A-action $\cdot$ as follows:

$$\pi \cdot atm(a) = atm(\pi(a)), \; \pi \cdot set(X) = set(\{\pi \cdot x \mid x \in X\}).$$

Let us consider:

- $FM_0(A) = \emptyset$;

- $FM_{\alpha+1}(A) = A + \wp_{fs}(FM_\alpha(A))$ (whenever the ordinal α has a successor);

- $FM_\lambda(A) = \underset{\alpha<\lambda}{\cup} FM_\alpha(A)$ (if λ is a limit ordinal).

where $+$ represents again the disjoint union of sets defined in Example 1(7). From Example 1, each $FM_\alpha(A)$ is an invariant set. The union of all $FM_\alpha(A)$ is called the *Fraenkel-Mostowski universe* and is denoted by $FM(A)$. We note that $FM(A)$ is a ZFA class, but we will refer to it as a (large) ZFA set because we will actually take count only on the (properties of the) group action with who it is equipped. Furthermore, the algebraic properties of sets equipped with permutation actions can be translated into algebraic properties of classes equipped with permutation actions.

We have that every element x of $FM(A)$ is either of the form $atm(a)$ with $a \in A$, or of the form $set(X)$ where X is a finitely supported set formed at an earlier ordinal stage than x. We call FM-sets the elements of the form $set(X)$, and atoms the elements of the form $atm(a)$. The FM universe $FM(A)$ is a subset (more exactly, a subclass) of $\nu(A)$. According to Proposition 1, the S_A-action $\cdot$ on $\nu(A)$ leads to an S_A-action on $FM(A)$. Thus, $FM(A)$ is an S_A-set together with the S_A-action $\cdot$ (it is actually a 'large' S_A-set, i.e. an S_A-class having the same properties as an S_A-set). Furthermore, $FM(A)$ is an invariant set because it contains only finitely supported elements with respect to $\cdot$.

An element $x \in \nu(A)$ that is not an atom is an FM-set (i.e. $x \in FM(A) \setminus A$) if and only if the following conditions are satisfied:

- y is an FM-set or an atom for all $y \in x$ (i.e. all the elements of x are hereditary finitely supported as elements in $\nu(A)$), and

- x has finite support with respect to the action $\cdot$.

An FM-set is actually an hereditary finitely supported subset of the invariant set $FM(A)$. An FM set X is not itself closed under the S_A-action $\cdot$ defined on $FM(A)$, unless $supp(X) = \emptyset$. Thus, an FM set is not necessarily equivariant in $FM(A)$ in the sense of Definition 2. This means that the restriction on a certain FM set X of the S_A-action $\cdot$ on $FM(A)$ does not necessarily lead to a new group action of S_A on X (since the codomain of the function $\cdot|_X$ is not necessarily X). Only an FM set with empty support is itself closed under the restriction of the S_A-action $\cdot$ on it. According to these remarks, and because invariant sets need to be closed under the

actions they are equipped (meaning that 'being an invariant set' means 'being an equivariant element at the following order stage in the hierarchical construction'), the invariant (or nominal) sets in the FM cumulative hierarchy are defined as those equivariant (i.e. empty supported) elements of $FM(A)$. This means that an FM set X is invariant if and only if the restriction $\cdot|_X$ of $\cdot$ on X is itself an S_A-action on X in the sense of Definition 2(1).

We provide an axiomatic presentation of FM set theory. The axioms are the ZFA axioms [14] together with the additional axiom of finite support (axiom 11).

Definition 6 *The next axioms characterize Fraenkel-Mostowski set theory:*

1. $\forall x.(\exists y.y \in x) \Rightarrow x \notin A$ (only non-atoms can have elements)

2. $\forall x, y.(x \notin A \, and \, y \notin A \, and \, \forall z.(z \in x \Leftrightarrow z \in y)) \Rightarrow x = y$
         (modified ZF axiom of extensionality to allow atoms)

3. $\forall x, y.\exists z.z = \{x, y\}$ (axiom of pairing)

4. $\forall x.\exists y.y = \{z \mid z \subseteq x\}$ (axiom of powerset)

5. $\forall x.\exists y.y \notin A \, and \, y = \{z \mid \exists w.(z \in w \, and \, w \in x)\}$ (axiom of union)

6. $\forall x.\exists y.(y \notin A \, and \, y = \{f(z) \mid z \in x\})$,
 for each functional formula $f(z)$ (axiom of replacement)

7. $\forall x.\exists y.(y \notin A \, and \, y = \{z \mid z \in x \, and \, p(z)\})$,
 for each formula $p(z)$ (axiom of separation)

8. $(\forall x.(\forall y \in x.p(y)) \Rightarrow p(x)) \Rightarrow \forall x.p(x)$ (induction principle)

9. $\exists x.(\emptyset \in x \, and \, (\forall y.y \in x \Rightarrow y \cup \{y\} \in x))$ (axiom of infinity)

10. *A is not finite.*

11. $\forall x.\exists S \subset A. \, S \, is \, finite \, and \, S \, supports \, x.$ (finite support axiom)

It is easy to see that $FM(A)$ is a model of FM set theory. Since the FM sets can be represented as (hereditary) finitely supported subsets of invariant sets, the inconsistency results presented below will be also valid in the framework of FM sets. The related inconsistency results would be obviously valid in the framework of nominal sets (which is a particular case of FSM when A is assumed to be a countable ZF set).

5   Choice Principles in Finitely Supported Mathematics

The validity of choice principles in various models ZF and of ZFA (including the permutation models) was studied in the last century. Since FSM is connected to the related permutation models, it became an open problem to study the consistency of choice principles with FSM.

The ZF choice principles are deeply described in [12].

- **Zorn's lemma (ZL):** Any non-empty inductive poset P (i.e. any poset P for which every totally ordered subset of P has an upper bound in P) has a maximal element;

- **Axiom of dependent choice (DC):** let R be a non-empty relation on a set X with the property that for each $x \in X$ there exists $y \in X$ with xRy. Then there exists a function $f : \omega \to X$ such that $f(n)Rf(n+1)$, $\forall n \in \omega$;

- **Axiom of countable choice (CC):** given any countable family (sequence) of non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$;

- **Axiom of partial countable choice (PCC):** given any countable family (sequence) of non-empty sets $\mathcal{F} = (X_n)_{n \in \mathbb{N}}$, there exists an infinite subset M of $\mathbb{N}$ such that it is possible to select a single element from each member of the family $(X_m)_{m \in M}$;

- **Axiom of choice over finite sets (AC(fin)):** given any family of finite non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$;

- **Axiom of countable choice over finite sets (CC(fin)):** given any countable family (sequence) of finite non-empty sets $\mathcal{F}$, it is possible to select a single element from each member of $\mathcal{F}$;

- **Boolean prime ideal theorem (PIT):** every Boolean algebra with $0 \neq 1$ has a maximal ideal (and hence a prime ideal);

- **Boolean ultrafilter theorem (UFT):** in a Boolean algebra, every filter can be enlarged to a maximal one;

- **Kinna-Wagner Selection Principle (KW)**: given any family $\mathcal{F}$ of sets of cardinality at least 2, there exists a function f on $\mathcal{F}$ such that $f(X)$ is a non-empty proper subset of X for each $X \in \mathcal{F}$;

- **Ordering principle (OP)**: every set can be totally ordered;

- **Order extension principle (OEP)**: every partial order relation on a set can be enlarged to a total order relation;

- **Axiom of Dedekind infiniteness (Fin)**: every infinite set X allows an injection $i : \mathbb{N} \to X$.

- **Surjection inverse principle (SIP)**: Any surjective mapping has a right inverse.

- **Finite power set equippolence principle (FPE)**: for every infinite set X there exists a bijection between X and $\wp_{fin}(X)$.

- **The generalized continuum hypothesis (GCH)**: states that if an infinite set Y is placed between an infinite set X and the power set of X, then Y either has the same cardinality as X or the same cardinality as the power set of X.

According to [12], the following implications are valid in ZF set theory:

1. **GCH $\Rightarrow$ AC $\Leftrightarrow$ SIP $\Leftrightarrow$ ZL $\Leftrightarrow$ FPE $\Rightarrow$ DC $\Rightarrow$ CC $\Leftrightarrow$ PCC $\Rightarrow$ CC(fin)**;

2. **PIT $\Leftrightarrow$ UFT**;

3. **AC $\Rightarrow$ UFT $\Rightarrow$ OEP $\Rightarrow$ OP $\Rightarrow$ AC(fin) $\Rightarrow$ CC(fin)**;

4. **CC $\Rightarrow$ Fin $\Leftrightarrow$ CC(fin)**;

5. **KW $\Rightarrow$ OP**.

Furthermore, all the previously mentioned choice principles are independent from the standard axioms of the ZF set theory [14], meaning that, for each of the above choice principles, both it and its negation are consistent within ZF. However, we should remark that the ZF relationship results are not necessarily valid in FSM (nor even in ZFA, as we remarked in Section 3). When we work in FSM we cannot employ a ZF theorem (or a ZFA theorem) in order to prove that a certain choice principle is consistent (i.e. it does

not lead to a contradiction) or not (i.e. its negation is provable). This is because all FSM relationship results between various choice principles have to be independently proved according to the finite support requirement.

The choice principles presented above can be formulated in FSM by requiring that all the constructions which appear in their statement are finitely supported (in order to be consistent with the FSM principle of finite support). In [1] we presented an FSM preliminary relationship result between several FSM choice principles which allow us to establish their inconsistency after a highly intricate development. In this section we consider/study more choice principles than in [1] and provide an (easier) independent proof (different from the proofs presented in [1]) for the inconsistency of each of them in FSM.

- **AC** has the form "Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finitely supported subsets of X, there exists a finitely supported choice function on $\mathcal{F}$."

- **ZL** has the form "Let P be a finitely supported subset of a non-empty invariant partially ordered set (i.e. a finitely supported subset of a non-empty invariant set equipped with an equivariant order relation) with the property that every finitely supported totally ordered subset of P has an upper bound in P. Then P has a maximal element."

- **DC** has the form "Let R be a non-empty finitely supported relation on a finitely supported subset Y of an invariant set X having the property that for each $x \in Y$ there exists $y \in Y$ with xRy. Then there exists a finitely supported function $f : \omega \to Y$ such that $f(n)Rf(n+1)$, $\forall n \in \omega$."

- **CC** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_{n \in \mathbb{N}}$ of subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists a finitely supported choice function on $\mathcal{F}$."

- **PCC** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_{n \in \mathbb{N}}$ of subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists an infinite subset M of $\mathbb{N}$ with the property that there is a finitely supported choice function on $(X_m)_{m \in M}$."

- **AC(fin)** has the form "Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finite subsets of X, there exists a

finitely supported choice function on $\mathcal{F}$."

- **CC(fin)** has the form "Given any invariant set X, and any countable family $\mathcal{F} = (X_n)_{n \in \mathbb{N}}$ of finite subsets of X such that the mapping $n \mapsto X_n$ is finitely supported, there exists a finitely supported choice function on $\mathcal{F}$."

- **PIT** has the form "Every finitely supported Boolean subalgebra of an invariant Boolean algebra (i.e. every finitely supported subset L' of an invariant set $(L, \cdot)$ endowed with an equivariant lattice order $\sqsubseteq$ on L and with the additional condition that L and L' are distributive and uniquely complemented) with $0 \neq 1$ has a maximal finitely supported ideal."

- **UFT** has the form "Any finitely supported filter of a finitely supported Boolean subalgebra of an invariant Boolean algebra can be extended to a finitely supported ultrafilter."

- **KW** has the form "Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finitely supported subsets of X of cardinality at least 2, there exists a finitely supported function f on $\mathcal{F}$ such that $f(Y)$ is a proper non-empty subset of Y for each $Y \in \mathcal{F}$."

- **OP** has the form "For every finitely supported subset X of an invariant set there exists a finitely supported total order relation on X."

- **OEP** has the form "Every finitely supported partial order relation on a finitely supported subset of an invariant set can be enlarged to a finitely supported total order relation."

- **Fin** has the form "Given any infinite finitely supported subset X of an invariant set, there exists a finitely supported injection from $\mathbb{N}$ to X."

- **SIP** has the form 'Given a finitely supported subset X of an invariant set, a finitely supported subset Y of an invariant set, and a finitely supported surjection from X onto Y, there exists a finitely supported right inverse of f'.

- **FPE** has the form 'For every infinite finitely supported subset X of an invariant set there exists a finitely supported bijection between X and $\wp_{fin}(X)$'.

- **GCH** has the from 'Let X be finitely supported subset of an invariant set. If Y is an infinite finitely supported subset of an invariant set having the property that there is a finitely supported injection from X to Y and a finitely supported injection from Y to $\wp_{fs}(X)$, then there exist either a finitely supported bijection from X to Y or a finitely supported bijection from Y to $\wp_{fs}(X)$.'

We say that a choice principle, rephrased in terms of finitely supported objects as above is consistent with (respect to) FSM (axioms) if it does not introduce a contradiction, i.e. if it is possible for it to be true in FSM. A related choice principle is inconsistent (it is not consistent) with (respect to) FSM (axioms) if it is not possible to the true in FSM, i.e. if its negation can be proved using FSM axioms. Although the above choice principles are independent from ZF axioms (meaning that their non-atomic formulations are valid in some models of ZF, while the negations of these non-atomic formulations are valid in other models of ZF), we prove below that all of them are in contradiction with the finite support requirement from FSM when atomic sets (equipped with the canonical permutation actions) are involved. For each of the above choice principles we construct finitely supported atomic sets for which the related choice principle fail.

Theorem 1 *None of the choice principles **AC, ZL, DC, CC, PCC, AC(fin), Fin, PIT, UFT, OP, KW, OEP, SIP, FPE**, and **GCH** (expressed in respect of the finite support principle as above) is consistent with Finitely Supported Mathematics, meaning that the negation of each of the above principles is a logical consequence of FSM.*

Proof: First we remark that the ZF relationship results between choice principles do not hold in FSM unless we are able to reprove them in terms of finitely supported objects, and so each choice principle must be analyzed separately in FSM. Thus, we will not use ZF relationship results between choice principles in order to prove the theorem.

We prove by contradiction that each of the above choice principles is inconsistent with FSM, meaning that the negation of each of them is provable in FSM when the finite support requirement is involved. So, for each choice principle, by presenting relevant counterexamples, we prove that if we assume that the choice principle is true in FSM, then we get a contradiction.

- Suppose **KW** holds in FSM. Consider the invariant set $(A, \cdot)$ of atoms equipped with the canonical action of the group of permutations of atoms $(\pi, a) \mapsto \pi \cdot a \stackrel{def}{=} \pi(a)$. Each permutation of atoms acts on a subset X of A by considering the pointwise action of that permutation on each atom from X. Particularly, we consider $\mathcal{F} := \{X \mid X \subset A,\, X\, finite,\, |X| = 2\}$ the family of (finitely supported) 2-element sets of atoms. $\mathcal{F}$ together with the action of the group of permutations of atoms $(\pi, \{x, y\}) \mapsto \pi \star \{x, y\} \stackrel{def}{=} \{\pi(x), \pi(y)\}$ is itself an invariant set (more exactly, it is an equivariant subset of the invariant set of all finitely supported subsets of A). Let f be a finitely supported Kinna-Wagner selection function on $\mathcal{F}$. Let S be a finite set (of atoms) supporting f. We may select a pair $Y := \{a, b\}$ from $\mathcal{F}$ such that a and b does not belong to S. This is because A is infinite, while $S \subseteq A$ is finite. Let π be a permutation of atoms which fixes S pointwise, and interchanges a and b. Since f satisfies the property that $f(X)$ is a non-empty proper (one-size) subset of X for all $X \in \mathcal{F}$, we have $f(Y) = \{a\}$ or $f(Y) = \{b\}$. Since π interchanges a and b, we have $\pi \star f(Y) = \pi(f(Y)) \neq f(Y)$ (where $\star$ represents the canonical action on $\wp_{fs}(A)$). However, $\pi \star Y = \{\pi(a), \pi(b)\} = \{b, a\} = Y$. Since π fixes S pointwise and S supports f, according to Proposition 3 we have $\pi(f(Y)) = \pi \star f(Y) = f(\pi \star Y) = f(Y)$, a contradiction. Thus, **KW** is inconsistent with FSM. In the same way, we remark that **AC(2)** (the axiom of choice for infinite families of 2-size subsets of an invariant set) is inconsistent with FSM (because the family of one-size subsets of A, emphasized in the proof of the inconsistency of **KW** as containing the codomain of the KW selection function f, can be identified with A), and so neither **AC(fin)** is valid in FSM. This means the negations of **KW** and **AC(fin)** are logical consequences of the finite support requirement in FSM.

- Suppose **OP** is true in FSM, and so there exists a finitely supported total order relation $<$ on the invariant set A. Let S be a finite set supporting $<$ and, $a, b, c \in A \setminus S$ with $a < b$. Since $(a\,c)$ fixes S pointwise and $<$ is a subset of the Cartesian product $A \times A$ supported by S, we have $(a\,c)(a) < (a\,c)(b)$, that is, $c < b$. However, we also have that $(a\,b)$ and $(b\,c)$ fixes S pointwise, and so $((a\,b) \circ (b\,c))(a) < ((a\,b) \circ (b\,c))(b)$, that is $b < c$. We get a contradiction, and so **OP** is not valid in FSM.

- Suppose that **OEP** holds in FSM. Then the equivariant partial order

relation $\leq$ on the invariant set A, defined by $x \leq y$ if and only if $x = y$ can be enlarged to a finitely supported total order relation on A. However, form the above item there does not exist a finitely supported total order relation on A.

- We prove now that the negation of **ZL** holds in FSM. More exactly, we prove that the family of all finite subsets of A, $(\wp_{fin}(A), \star, \subseteq)$ is an invariant set having the property that any finitely supported totally ordered subset of $\wp_{fin}(A)$ has an upper bound in $\wp_{fin}(A)$, but $\wp_{fin}(A)$ does not have a maximal element. First we remark that $\wp_{fin}(A)$ is an invariant set together with the permutation action $(\pi, X) \mapsto \pi \star X \overset{def}{=} \{\pi \cdot a \mid a \in X\} = \{\pi(a) \mid a \in X\}$. The relation $\subseteq$ on $\wp_{fin}(A)$ is equivariant because whenever $X \subseteq Y$ we have $\pi \star X \subseteq \pi \star Y$ for all $\pi \in S_A$.

  Let $\mathcal{F}$ be a finitely supported totally ordered subset of the invariant set $\wp_{fin}(A)$. Since $\mathcal{F}$ is totally ordered with respect to the inclusion relation on $\wp_{fin}(A)$, then there does not exist two different finite subsets of A of the same cardinality belonging to $\mathcal{F}$. Since $\mathcal{F}$ is finitely supported, then there exists a finite set $S \subseteq A$ such that $\pi \star Y \in \mathcal{F}$ for each $Y \in \mathcal{F}$ and each π that fixes S pointwise. However, since permutations of atoms are bijective, for each $Y \in \mathcal{F}$ and each permutation of atoms π we have that the cardinality of $\pi \star Y$ coincides with the cardinality of Y. Since there does not exist two distinct elements in $\mathcal{F}$ having the same cardinality, we conclude that $\pi \star Y = Y$ for all $Y \in \mathcal{F}$ and all π fixing S pointwise. Thus, $\mathcal{F}$ is uniformly supported by S (i.e. all the elements of $\mathcal{F}$ are supported by the same set S). However, there could exist only finitely many finite subsets of A (i.e. only finitely many elements in $\wp_{fin}(A)$, and hence only finitely many elements in $\mathcal{F}$) supported by S, namely the subsets of S. Thus, $\mathcal{F}$ must be finite, and so there exists the (finite) union of the members of $\mathcal{F}$ which is an elements of $\wp_{fin}(A)$ (expressed as a finite union of finite sets) and an upper bound for $\mathcal{F}$. Suppose by contradiction that there exists a maximal element X_0 of $\wp_{fin}(A)$. Then $X_0 = \{a_1, \ldots a_n\}$, $a_1, \ldots, a_n \in A$ for some $n \in \mathbb{N}$. Since A is infinite, there exists $b \in A \setminus X_0$. However, $\{a_1, \ldots a_n\} \subsetneq \{a_1, \ldots a_n, b\}$ with $\{a_1, \ldots a_n, b\} \in \wp_{fin}(A)$, which contradicts the maximality of X_0.

- Let us assume that **Fin** is true in FSM. Thus, we can find a finitely supported injection $f : \mathbb{N} \to A$. Let us consider $m, n \in \mathbb{N}$ such that

$m \neq n$ and $f(m), f(n) \notin supp(f)$. These atoms exist because $f(\mathbb{N}) \subseteq A$ is infinite, while $supp(f) \subseteq A$ is finite. Hence $(f(m) f(n)) \star f = f$, where $\star$ is defined as in Proposition 2. Let us denote $(f(m) f(n))$ by π. Since the single possible S_A-action $\diamond$ on $\mathbb{N}$ is the trivial one (defined in Example 1(3)), according to Proposition 2, we have $f(m) = (\pi \star f)(m) = \pi(f(\pi^{-1} \diamond m)) = \pi(f(m)) = f(n)$ which contradicts the injectivity of f. Thus, **Fin** is not valid when the finite support principle from FSM is involved.

- From Proposition 5.2.2 in [18] (whose proof remains valid even when the set of atoms is not countable), there exists an invariant Boolean algebra having a finitely supported filter that cannot be extended to a finitely supported ultrafilter. Therefore, **UFT** fails in the framework of invariant sets.

  Alternatively, the proof of Theorem 4.39 from [12] can be reformulated in FSM because, with the notations in [12], each finite subset F of X is obviously finitely supported, and the mappings $F \mapsto X_F$ and $(E, F) \mapsto A_{(E,F)}$ are finitely supported by $supp(F) \cup supp(R)$ and $supp(E) \cup supp(F) \cup supp(R)$, respectively; it follows that **UFT** $\Rightarrow$ **OEP** holds in FSM. Since the negation of **OEP** is provable in FSM, we have that the negation of **UFT** is provable in FSM.

- By contradiction, assume that the choice principle **PIT** holds in FSM. Thus, every invariant Boolean algebra with $0 \neq 1$ has a maximal finitely supported ideal, and hence a maximal finitely supported filter. We prove that any equivariant filter of an arbitrary invariant Boolean algebra can be extended to a finitely supported maximal filter. Indeed, consider an invariant Boolean algebra $(B, \wedge, \vee, \cdot)$ and let $\mathcal{F}$ be an equivariant filter in B. Therefore, $\mathcal{F}' = \{x' \mid x \in \mathcal{F}\}$ (where x' represents the complement of x) is a equivariant ideal in B. We define the relation $\sim_{\mathcal{F}'}$ on B by $x \sim_{\mathcal{F}'} y$ if and only if $(x \wedge y') \vee (y \wedge x') \in \mathcal{F}'$. Since the operations $\wedge$, $\vee$ and *complement* are all equivariant functions (according to the equivariance principle in FSM), and because $\mathcal{F}'$ is an equivariant subset of B, it follows that $\sim_{\mathcal{F}'}$ is also an equivariant subset of $B \times B$. Moreover, the quotient lattice $B/\mathcal{F} \stackrel{def}{=} B/\sim_{\mathcal{F}'}$ is an invariant set (with the S_A-action $\star$ defined by $\pi \star [x]_{\sim_{\mathcal{F}'}} = [\pi \cdot x]_{\sim_{\mathcal{F}'}}$ for all $\pi \in S_A, x \in B$). Thus, because $(B/\mathcal{F}, \bar{\wedge}, \bar{\vee})$ is also a Boolean algebra (according to the general theory of Boolean algebras), from Proposition 3, it follows that $(B/\mathcal{F}, \bar{\wedge}, \bar{\vee})$ is an invariant Boolean

algebra, where the equivariant operations $\bar{\wedge}, \bar{\vee}$ on $B/\mathcal{F}$ are defined by $[x]_{\sim_{\mathcal{F}'}} \bar{\wedge} [y]_{\sim_{\mathcal{F}'}} = [x \wedge y]_{\sim_{\mathcal{F}'}}$ and $[x]_{\sim_{\mathcal{F}'}} \bar{\vee} [y]_{\sim_{\mathcal{F}'}} = [x \vee y]_{\sim_{\mathcal{F}'}}$, for all $[x]_{\sim_{\mathcal{F}'}}, [y]_{\sim_{\mathcal{F}'}} \in B/\mathcal{F}$. According to **PIT**, there exists a finitely supported maximal filter G in $B/\mathcal{F}$. Consider the natural map from B onto the corresponding quotient space, $f : B \to B/\mathcal{F}$ defined by $f(x) = [x]_{\sim_{\mathcal{F}'}}$, for all $x \in B$. By its definition, we have that f is equivariant. According to Proposition 3, we have that $f^{-1}(G)$ is finitely supported by $supp(f) \cup supp(G) = \emptyset \cup supp(G) = supp(G)$. Moreover, $f^{-1}(G)$ is a maximal filter in B such that $\mathcal{F} \subseteq f^{-1}(G)$. Thus, $f^{-1}(G)$ is a finitely supported maximal filter in B that enlarges $\mathcal{F}$.

However, as it is proved in Proposition 5.2.2 of [18], there exists an invariant Boolean algebra having an equivariant filter that cannot be extended to a finitely supported ultrafilter. The related filter is the filter f defined on page 151 in [18]. We obtain a contradiction, and so **PIT** fails in FSM.

- The choice principles **CC** and **PCC** are in contradiction with the finite support principle from FSM. We consider the countable family $(X_n)_{n\in\mathbb{N}}$ where X_n is the set of all injective n-tuples from A. Since A is infinite, it follows that each X_n is non-empty. In FSM, each X_n is equivariant because A is an invariant set and each permutation is a bijective function; more exactly, the image an injective n-tuple of atoms under an arbitrary permutation $\pi \in S_A$ is another injective n-tuple of atoms. Therefore, the family $(X_n)_{n\in\mathbb{N}}$ is equivariant and the mapping $n \mapsto X_n$ is also equivariant.

If we assume that **CC** is true, then according to the formulation of **CC** in FSM, there exists a finitely supported choice function f on $(X_n)_{n\in\mathbb{N}}$. Let $f(X_n) = y_n$ with each $y_n \in X_n$. Let $\pi \in Fix(supp(f))$. According to Proposition 4, and because each element X_n is equivariant according to its definition, we obtain that $\pi \cdot y_n = \pi \cdot f(X_n) = f(\pi \star X_n) = f(X_n) = y_n$, where by $\star$ we denoted the S_A-action on $(X_n)_{n\in\mathbb{N}}$ and by $\cdot$ we denoted the S_A-action on $\cup_n X_n$. Therefore, each element y_n is supported by $supp(f)$. However, since each y_n is a finite tuple of atoms, we have $supp(y_n) = y_n$, $\forall n \in \mathbb{N}$. Since $supp(y_n) \subseteq supp(f)$, $\forall n \in \mathbb{N}$, we obtain $y_n \subseteq supp(f)$, $\forall n \in \mathbb{N}$. Since each y_n has exactly n elements, this contradicts the finiteness of $supp(f)$.

If we assume that **PCC** is true, then according to the formulation of **PCC** in FSM, there exists an infinite subset M of $\mathbb{N}$ and a finitely

supported choice function g on $(X_m)_{m \in M}$. Let $g(X_m) = y_m$ with each $y_m \in X_m$. As in the paragraph above we obtain $y_m \subseteq supp(g)$ for all $m \in M$. Since y_m has exactly m elements for each $m \in M$, and since M is infinite, we contradict the finiteness of $supp(g)$.

- We prove that the choice principle **DC** fails in FSM. Assume, by contradiction, that **DC** is true in FSM. Let us consider the invariant set $(\wp_{fin}(A), \star)$. The relation $R \stackrel{def}{=} \subsetneq$ defined on $\wp_{fin}(A)$ is equivariant because whenever $X \subsetneq Y$ we have $\pi \star X = \{\pi(a) \mid a \in X\} \subsetneq \{\pi(a) \mid a \in Y\}$ $= \pi \star Y$ for all $\pi \in S_A$. Let $X \in \wp_{fin}(A)$. Then $X = \{a_1, \ldots a_n\}$, $a_1, \ldots, a_n \in A$ for some $n \in \mathbb{N}$. Since A is infinite, there exists $b \in A \backslash X$, and so $X = \{a_1, \ldots a_n\} \subsetneq \{a_1, \ldots a_n, b\} \stackrel{def}{=} Y$ with $Y \in \wp_{fin}(A)$. Thus, for all $X \in \wp_{fin}(A)$ there is $Y \in \wp_{fin}(A)$ such that $X R Y$. Then, from **DC**, there exists a finitely supported function $f : \mathbb{N} \to \wp_{fin}(A)$ such that $f(n) R f(n+1)$, i.e. a finitely supported function $f : \mathbb{N} \to \wp_{fin}(A)$ with the property that $f(n) \subsetneq f(n+1)$ for all $n \in \mathbb{N}$. Then f is injective. Thus, $f(\mathbb{N})$ is an infinite family of finite subsets of A (a infinite family of elements in $\wp_{fin}(A)$) with the property that all the elements of $f(\mathbb{N})$ are supported by the same finite set $supp(f)$. This is because, according to Proposition 4, $\pi \star f(n) = f(\pi \diamond n) = f(n)$ for all $n \in \mathbb{N}$ and all $\pi \in Fix(supp(f))$, where $\diamond$ represents the trivial S_A-action on $\mathbb{N}$. However, there are only finitely many elements of $\wp_{fin}(A)$ supported by $supp(f)$, namely the subsets of $supp(f)$. Therefore, **DC** fails in FSM.

- We prove that **SIP** fails in FSM. Let us consider the invariant set $(\wp_{fs}(A), \star)$ and the trivial invariant set $(\mathbb{N}, \diamond)$. Define $f : \wp_{fs}(A) \to \mathbb{N}$ by $f(X) = |supp(X)|$ for any $X \in \wp_{fs}(A)$, where $|supp(X)|$ represents the number of elements of $supp(X)$. Let us consider the equivariant family $(X_i)_{i \geq 1}$ where each X_i is the set of all i-size subsets of A. Since A is infinite, it follows that each $X_i, i \geq 1$ is non-empty. Furthermore, for any fixed $n \in \mathbb{N}$ we have $f(\{x_1, \ldots, x_n\}) = |supp(\{x_1, \ldots, x_n\})| = n, \forall \{x_1, \ldots, x_n\} \in X_n$ (see Example 1(5)), and so the image of f under X_n is $f(X_n) = n$ for all $n \in \mathbb{N}$. Thus, $\mathbb{N} = f(\underset{i \in \mathbb{N}}{\cup} X_i) \subseteq f(\wp_{fs}(A)) = Im(f)$, and so f is surjective. No choice principle is required because, for proving the surjectivity of f, we do not need to identify a set of representatives for the family $(X_i)_{i \geq 1}$. Now we prove that f is equivariant. According to Proposition 1 and because any permutation of atoms is one-to-one, we have $f(\pi \star X) = |supp(\pi \star X)| =$

$|\pi(supp(X))| = |supp(X)| = f(X) = \pi \diamond f(X)$ for all $\pi \in S_A$ and $X \in \wp_{fs}(A)$. From Proposition 4, f is equivariant. Suppose, by contradiction, that **SIP** is true in FSM. Then f has a finitely supported right inverse g, that is, $f \circ g = 1_{\mathbb{N}}$. Therefore, $g : \mathbb{N} \to \wp_{fs}(A)$ is injective. Then $g(\mathbb{N})$ is an infinite family of subsets of A with the property that all the elements of $g(\mathbb{N})$ are supported by the same finite set $supp(g)$. This is because $\pi \star g(n) = g(\pi \diamond n) = g(n)$ for all $n \in \mathbb{N}$ and all $\pi \in Fix(supp(g))$. However, there are only finitely many subsets of A supported by $supp(g)$, namely the subsets of $supp(g)$ and the supersets of $A \setminus supp(g)$, and so we get a contradiction.

- We prove that **FPE** fails in FSM. For this we consider the invariant sets $(A, \cdot)$ and $(\wp_{fin}(A), \star)$ and we prove that there does not exist a finitely supported surjection from A onto $\wp_{fin}(A)$. Suppose by contradiction that there is a finitely supported surjection $f : A \to \wp_{fin}(A)$. Let us fix an element $a \in A$ with $a \notin supp(f)$. Let b be an arbitrary element from $A \setminus supp(f)$. Since $b \notin supp(f)$, we have that the transposition $(a\,b)$ fixes every element from $supp(f)$. However, $supp(f)$ supports f, and so according to Proposition 3, we have $f((a\,b)(c)) = (a\,b) \star f(c)$ for all $c \in A$. In particular, $f(b) = f((a\,b)(a)) = (a\,b) \star f(a)$. If $f(a)$ is an n-size subset of A, then, because transpositions are injective functions, $f(b)$ is another n-size subset of A. Thus, because b has been chosen arbitrarily from $A \setminus supp(f)$, we have $|f(a)| = |f(b)|$ for all $a, b \in A \setminus supp(f)$. However, $supp(f)$ is finite. If we assume $|supp(f)| = k$ (k is finite), then $Im(f)$ can contain subsets of A of at most (finite) $k + 1$ different sizes. This means f cannot be surjective.

- We prove that **GCH** fails in FSM, meaning that there exists an invariant set X and an invariant set Y placed between X and $\wp_{fs}(X)$ such that there is no finitely supported bijection between Y and X and no finitely supported bijection between Y and $\wp_{fs}(X)$. Let us consider $X = A$ and $Y = \wp_{fin}(A)$. The mapping $i : A \to \wp_{fin}(A)$ defined by $i(x) = \{x\}$ is obviously an equivariant injective mapping from $(A, \cdot)$ to $(\wp_{fin}(A), \star)$. However, from the above item there does not exist a finitely supported surjection from A onto $\wp_{fin}(A)$.

  We obviously have the equivariant identity injection from $\wp_{fin}(A)$ to $\wp_{fs}(A)$. We prove below that there does not exist a finitely supported injective mapping from $\wp_{fs}(A)$ onto one of its finitely supported proper subsets, i.e. any finitely supported injection $f : \wp_{fs}(A) \to \wp_{fs}(A)$ is

also surjective. Let us consider a finitely supported injection $f :$ $\wp_{fs}(A) \to \wp_{fs}(A)$. Suppose, by contradiction, $Im(f) \subsetneq \wp_{fs}(A)$. This means that there exists $X_0 \in \wp_{fs}(A)$ such that $X_0 \notin Im(f)$. Since f is injective, we can define an infinite sequence $\mathcal{F} = (X_n)_{n \in \mathbb{N}}$ starting from X_0, with distinct terms of form $X_{n+1} = f(X_n)$ for all $n \in \mathbb{N}$. Furthermore, according to Proposition 4, for a fixed $k \in \mathbb{N}$ and $\pi \in Fix(supp(f) \cup supp(X_k))$, we have $\pi \star X_{k+1} = \pi \star f(X_k) = f(\pi \star X_k) = f(X_k) = X_{k+1}$. Then, $supp(X_{n+1}) \subseteq supp(f) \cup supp(X_n)$ for all $n \in \mathbb{N}$, and by induction on n we have that $supp(X_n) \subseteq supp(f) \cup supp(X_0)$ for all $n \in \mathbb{N}$. We obtained that each element $X_n \in \mathcal{F}$ is supported by the same finite set $S := supp(f) \cup supp(X_0)$. However, there could exist only finitely many subsets of A (i.e. only finitely many elements in $\wp_{fs}(A)$) supported by S, namely the subsets of S and the supersets of $A \setminus S$ (where a superset of $A \setminus S$ is of form $A \setminus X$ with $X \subseteq S$). We contradict the statement that the infinite sequence $(X_n)_{n \in \mathbb{N}}$ never repeats. Thus, f is surjective, and so there could not exist a bijection between $\wp_{fin}(A)$ and $\wp_{fs}(A)$. $\square$

Remark 1 *According to [7], the following implication holds in the ZF framework:* **AC(fin)** *implies that "every infinite set X has an infinite subset Y such that $X \setminus Y$ is also infinite". A similar result hold for* **Fin** *according to [16]. However, we cannot directly conclude that such an implication holds in (can be reformulated in) FSM where only finitely supported objects are allowed. We cannot say (without a proof which is consistent with the finite support requirement from FSM) that the following statement is valid: " 'Given any invariant set X, and any finitely supported family $\mathcal{F}$ of non-empty finite subsets of X, there exists a finitely supported choice function on $\mathcal{F}$ (i.e.* **AC(Fin)** *in FSM)' implies 'Every infinite invariant set X has an infinite finitely supported subset Y such that $X \setminus Y$ is also finitely supported and infinite'." Therefore, we cannot directly conclude that* **AC(fin)** *is false in FSM just because the statement "Every infinite invariant set X has an infinite finitely supported subset Y such that $X \setminus Y$ is also finitely supported and infinite" is false in FSM. Such a result requires a separate proof reformulated in terms of finitely supported objects.*

More explicitly, an amorphous set in FSM is the set of atoms. However it is not an ordinary (non-atomic) ZF set, but an atomic one endowed with the canonical action $(\pi, a) \mapsto \pi(a)$. If we use, for example, (the proof of) Brunner's result in ZF (without reformulating it in FSM) [7] we would obtain that there exists an atomic ZF set not satisfying **AC(fin)**, *i.e. an atomic ZF*

*set X having an infinite family $\mathcal{F}$ of finite subsets with no choice function. However, we do not know if these X and $\mathcal{F}$ are finitely supported subsets of invariant sets (under the canonical action of S_A), and so we actually do not contradict immediately **AC(fin)** in FSM which should be valid only for finitely supported families (under the canonical hierarchical S_A-actions) and not for those atomic ZF families which are not finitely supported.*

In fact, Remark 1 states that we cannot prove an FSM result only by employing a ZF result (without an additional proof made according to the finite support requirement). More examples of results which are valid in the ZF framework, but fail in FSM are in Section 3.

Note that, according to the definition of an FM-set, the previous choice principles can as well be reformulated in terms of FM-sets by informally replacing "finitely supported subset of an invariant set" with "FM-set". For example, **AC** can be reformulated in the form: "Given any finitely supported family $\mathcal{F}$ of non-empty FM-sets, there exists a finitely supported choice function on $\mathcal{F}$", and so on. The non-validity of various choice principles in the Fraenkel-Mostowski cumulative universe (which is a model of axiomatic FM set theory) can be proved exactly as we proved the inconsistency of the related choice principles in FSM. Formally can express this as:

Corollary 1 *None of the choice principles **AC**, **ZL**, **DC**, **CC**, **PCC**, **AC(fin)**, **Fin**, **PIT**, **UFT**, **OP**, **KW**, **OEP**, **SIP**, **FPE** and **GCH** is consistent with the FM axiomatic set theory.*

Since nominal sets [19] are defined as invariant sets with the requirement that the set of atoms is countable, by particularizing the results in this paper for a countable infinite set of atoms, and by noting that in the proof of Theorem 1 the counter-examples constructed for proving the inconsistency of choice principles are actually represented by invariant sets and not by arbitrary finitely supported subsets of invariant sets, we obtain:

Corollary 2 *All the choice principles **AC**, **ZL**, **DC**, **CC**, **PCC**, **AC(fin)**, **Fin**, **PIT**, **UFT**, **OP**, **KW**, **OEP**, **SIP**, **FPE** and **GCH** are inconsistent in the framework of nominal sets.*

6   EFM Set Theory

According to the finite support axiom in FM set theory, each subset A has to be finitely supported, and so only finite or cofinite sets of atoms are

allowed in the FM universe. The finite support axiom of FM set theory is very strong, meaning it requires the existence of finite supports for any set theoretical construction. One can think to study the consequences of replacing this strong axiom with a more relaxed one stating only that each subset of A is either finite or cofinite. Obviously, "finite-cofinite" is easier to manipulate than "finitely supported". The aim of this approach is to replace the requirement "finite support for all sets (built on a cumulative hierarchy)" with "finite support only for subsets of atoms" in order to obtain some similar results as in the FM case. Extended Fraenkel-Mostowski axiomatic set theory (EFM) represents a refinement of FM set theory obtained by replacing the finite support axiom with a consequence of it requiring only an amorphousness structure of the set of atoms. Although the finite support axiom from FM set theory is relaxed in this new theory (i.e. we do not require that each bijection of A is finitely supported), many properties of the group of all bijections of A, such as torsioness or locally finiteness are preserved [1].

Definition 7 *The following axioms give a complete characterization of Extended Fraenkel-Mostowski set theory:*

1. *Axioms 1-10 are the same as in Definition 6.*

11' *Each subset of A is either finite or cofinite.   (axiom of the structure of A)*

A model of EFM set theory is represented by the slightly modified model $\nu(A)$ of ZFA set theory, where A is assumed to be an infinite amorphous set of atoms. More precisely, we define:

- $\mu_0(A) = \emptyset$;

- $\mu_{\alpha+1}(A) = A + \wp(\mu_\alpha(U))$, where $+$ denotes the disjoint union of sets;

- $\Gamma_{\alpha+1}(A) = \mu_{\alpha+1}(A) \setminus T_0$, where T_0 is formed by those ZFA subsets of A which are simultaneously infinite and coinfinite;

- $\Gamma_\lambda(A) = \underset{\alpha<\lambda}{\cup}(\nu_\alpha(A) \setminus T_0)$ (λ a limit ordinal).

Let $\Gamma(A)$ be the union of all $\Gamma_\alpha(A)$. Then $\Gamma(A)$ is a model of EFM set theory. The EFM sets are not necessarily finitely supported. For example $\wp(\wp_{fin}(A) \cup \wp_{cofin}(A))$ is a well defined EFM set, but it is not and FSM set (since it contains non-finitely supported elements); only $\wp_{fs}(\wp_{fin}(A) \cup \wp_{cofin}(A)) = \wp_{fs}(\wp_{fs}(A))$ is well defined in FSM.

7   Choice Principles in the EFM Framework

The ZF choice principles described in the first part of Section 5 can be directly reformulated in EFM by replacing 'ZF' with 'ZFA'.

Theorem 2 *The choice principles **Fin** and **AC(fin)** are inconsistent with the axioms of EFM set theory.*

Proof: According to [16], we know that **Fin** implies "Every infinite set X has an infinite subset Y such that $X \setminus Y$ is also infinite". According to [7], the following implication holds: **AC(fin)** implies "Every infinite set X has an infinite subset Y such that $X \setminus Y$ is also infinite". These results remain valid in ZFA set theory and in EFM set theory. This is becuase the presence of atoms in the EFM framework does not change their ZF proof, and because the finite support requirement is no longer mandatory in EFM set theory. Actually EFM set theory is the ZFA set theory where amorphous sets are allowed. According to axiom 11' in EFM set theory, for each subset X of the infinite set A we have that either X is finite or $A \setminus X$ is finite. Therefore, the statement "Every infinite set X has an infinite subset Y such that $X \setminus Y$ is also infinite" is false in the EFM framework. Thus, the choice principles **Fin** and **AC(fin)** fail in EFM set theory. The inconsistency of **Fin** with the EFM framework can also be proved directly from axiom 11'. We assume that **Fin** is a valid statement in the EFM framework. Then we can find an injection $f : \mathbb{N} \to A$. Obviously $f(2\mathbb{N})$ and $f(2\mathbb{N} + 1)$ are disjoint, infinite subsets of A. Therefore, $f(2\mathbb{N})$ is infinite and coinfinite, and this contradicts the structure of A. Thus, **Fin** fails in the EFM framework.   □

Remark 2 *Note that the proof of Theorem 2 is consistent with the EFM axioms. However, such a proof cannot be made in FSM because in FSM only finitely supported objects are allowed. Although axiom 11' is a direct consequence of axiom 11 of FM set theory, a result obtained in the EFM framework remains valid in the FM framework only if all the objects appearing in its proof are finitely supported according to canonical hierarchically defined S_A-actions. More exactly, the required implications in the proof of Theorem 2 are not necessarily valid in FSM, unless we can reformulate them in terms of finitely supported objects. A proof of an FSM result (and of an FM result) should involve only finitely supported constructions, i.e. it should be internally consistent in the world of finitely supported structures and not retrieved from ZF or ZFA. This is the reason why we do not present Theorem 1 as a consequence of Theorem 2. More details are in Remark 1.*

Actually we want to say that although A is amorphous in ZF (or in ZFA if A is associated with the set of atoms from ZFA), from Brunner's result [7] we would obtain that there exists an atomic ZF set X having an infinite family $\mathcal{F}$ of finite subsets with no choice function, but we would not be able to establish directly whether X and $\mathcal{F}$ are finitely supported under the canonical permutation action in order to obtain the inconsistency of $AC(fin)$ with FSM; however this reasoning works for establishing the inconsistency of $AC(fin)$ with RFM where the finite support principle is no longer required.

Corollary 3 *The choice principles AC, ZL, DC, CC, PCC PIT, UFT, OP, KW, OEP, SIP, FPE and GCH are inconsistent with the axioms of EFM set theory, meaning that the negation of each of the above principle is a logical consequence of EFM axioms.*

Proof: Since $AC(fin)$ and **Fin** are inconsistent with the axioms of EFM set theory and the ZF relationship results between choice principles can be adequately reformulated in EFM (see [14], [12] and [11]), the related choice principles are also inconsistent with the axioms of EFM set theory. $\square$

8 Conclusion and Further Work

FSM represents an appropriate framework to work with infinite atomic structures in terms of finitely supported objects. FSM is a mathematics which is consistent in respect of the finite support requirement stating that every logical construction involving elements from a previously fixed ZF set of basic elements A should be finitely supported under the hierarchically constructed canonical group action of the group of all permutations of A. FSM is actually a slightly modified algebraic version of nominal sets, reformulated for possibly non-countable sets of atoms (it treats nominal sets endowed with a "nominal" algebraic structure, such as monoids, groups, partially ordered sets etc). FSM is developed in the ZF framework, but it can be adequately reformulated when the fixed ZF set A considered in its construction is replaced by the set of atoms in ZFA. EFM set theory is actually the ZFA set theory where A is an infinite amorphous set.

The characterization of particular finitely supported algebraic structures (that are included in the general framework called FSM), using the 'nominal' approach, have already triggered significant applications. For example, a generalization of invariant monoids was used in order to characterize automata and languages over infinite alphabets [6]. Invariant partially ordered sets

were used to solve the Scott recursive domain equation $D \cong (D \to D)$ within invariant sets, and so to implement programming languages incorporating new facilities for manipulating syntax involving fresh names and binding operations [20]. Using an original result named 'the Tarski-like fixpoint theorem for invariant complete lattices' and some original properties of invariant Galois connections, we developed a theory of abstract interpretation of programming languages [2] within finitely supported structures. Some calculability properties (obtained from the narrowing and widening techniques of approximation) of the fixed points of a class of finitely supported functions were also proved in [2]. Since there exist invariant complete lattices failing to be ZF complete, in [2] it is proved that there may also exist abstract interpretations of some programming languages that can be easier described by using finitely supported sets. The theory of fuzzy sets was developed within finitely supported structures, by considering an equivariant association between a certain invariant set and a finitely supported membership function characterizing the degree on membership of each element in the invariant set [4]. More precisely, in this paper, we were able to describe in a computable manner the fuzzy membership functions defined on an infinite universe of discourse. Furthermore, we proved that the family of finitely supported fuzzy subsets of a certain invariant set is an invariant complete lattice although it not necessarily a full ZF complete lattice.

The goal of this paper was to present the existing development regarding FM sets, nominal sets and finitely supported algebraic structures, to emphasize the connections between these frameworks and other fundamental approaches in mathematics such as Tarski's concept of logicality, to provide new set theoretical properties (regarding the consistency of choice principles) with the framework of finitely supported algebraic structures, and to announce various properties regarding cardinalites and Dedekind finiteness. According to Theorem 1, the choice principles **AC**, **ZL**, **DC**, **CC**, **PCC**, **AC(fin)**, **Fin**, **PIT**, **UFT**, **OP**, **KW**, **OEP**, **SIP**, **FPE** and **GCH** are invalid in FSM. As a simple consequence (because FSM can be adequately reformulated over ZFA), it follows that these principles are also inconsistent with the axioms of FM set theory. Moreover, the inconsistency results presented in this paper are also valid in the framework of nominal sets developed in [19]. Thus, were also able to provide a study of the consistency of the choice principles for both Pitts nominal framework and Gabbay-Pitts axiomatic FM framework. According to Theorem 2 and Corollary 3, the previous choice principles are also inconsistent with the EFM axioms. Since

the theory of invariant sets makes sense even if the set of atoms is not countable, the inconsistency results in FSM presented in this paper (adequately reformulated when the fixed ZF set A is replaced by the set of atoms in ZFA), do not overlap on some related properties in the basic or in the second Fraenkel models of ZFA set theory which are defined using countable sets of atoms [14]. Also, the results in this paper do not follow immediately from [19] because in that paper the nominal sets are defined over countable sets of atoms, while we defined invariant sets over possible non-countable sets of atoms; in the viewpoint from [19] (i.e. if the set of atoms is countable) the inconsistency of the countable choice principles would be trivial. Since no information about the countability of the set of atoms is available in a general theory of invariant sets, the consistency of **CC(fin)** in FSM remains an open problem. Moreover, EFM set theory is itself inconsistent when the set of atoms is countable, because countable sets have both infinite and coinfinite subsets (contradicting axiom 11').

Finally, we note that we proved the non-validity of the previously mentioned choice principles in FSM by constructing counter-examples represented by finitely supported atomic sets not satisfying the related choice principles. We also mention that the non-atomic ZF sets are part of FSM (as they are trivial invariant sets hierarchically constructed over the non atomic $\emptyset$). Thus, the non-validity of the FSM atomic forms of the related choice principles does not affect the independence of their non-atomic forms in ZF (since we constructed relevant counter-examples by involving only structures that use atoms in their construction). This means no non-atomic ZF result is weakened under the FSM approach, but there may exist classical ZF results (i.e. results valid for trivial non-atomic sets) that may no longer remain valid when extending them from the framework of trivial invariant (non-atomic) structures to the framework of finitely supported atomic structures equipped with canonical permutation actions. Actually, the goal of this paper was to study if a certain ZF choice principle remain valid when translating it from the non-atomic ZF framework into the atomic FSM. When replacing 'usual non-atomic set' with 'finitely supported atomic set', all the previously mentioned choice principles, rephrased in terms of finitely supported objects, were proved to be invalid results (in contradiction with the finite support requirement), meaning that their negations are logical consequences of the FSM axioms. However, we have to mention that in the light of the S-finite support principle stated in Section 2, there exist many other ZF results, originally proved for non-atomic structures, which remain valid when atomic

structures with finite supports are involved (see Chapter 3 in [1] for relevant examples).

Future Developments. We intend to provide a full study regarding cardinalities and Dedekind-finiteness in Finitely Supported Mathematics. In particular, we enumerate the following theorems, also related to the consistency of choice principles, that hold in FSM. In the view of Section 3 their proofs are not trivial and will be the topic of future work.

Cardinalities:

Two FSM sets X and Y are called equipollent if there exists a finitely supported bijection $f : X \to Y$. The FSM cardinality of X is defined as the equivalence class of all FSM sets equipollent to X, and is denoted by $|X|$. This means that for two FSM sets X and Y we have $|X| = |Y|$ if and only if there exists a finitely supported bijection $f : X \to Y$. On the family of cardinalities we can define the relations:

- $\leq$ by: $|X| \leq |Y|$ if and only if there is a finitely supported injective mapping $f : X \to Y$.

- $\leq^*$ by: $|X| \leq^* |Y|$ if and only if there is a finitely supported surjective mapping $f : Y \to X$.

We claim that the following properties of $\leq$ and $\leq^*$ hold.

1. The relation $\leq$ is equivariant, reflexive, anti-symmetric and transitive, but it is not total.

2. The relation $\leq^*$ is equivariant, reflexive and transitive, but it is not anti-symmetric, nor total.

3. There exists an invariant set X (particularly the set A of atoms) having the following properties.

   - $|X \times X| \nleq |X|$;
   - $|X \times X| \nleq^* |X|$;
   - $|X \times X| \nleq |\wp_{fs}(X)|$;
   - $|X \times X| \nleq^* |\wp_{fs}(X)|$;
   - $|X| \lneq |\wp_n(X)| \lneq |\wp_{fs}(X)|$, where $\wp_n(X)$ is the family of all n-size subsets of X;
   - $|\wp_{fs}(X) \times \wp_{fs}(X)| \nleq |\wp_{fs}(X)|$;

- $|\wp_{fs}(X) \times \wp_{fs}(X)| \nleq^* |\wp_{fs}(X)|$;

Dedekind Finiteness:

An FSM set X is Dedekind finite if there does not exist a finitely supported injection of X onto one of its finitely supported proper subsets. Regarding Dedekind finite sets we claim that the following properties hold.

1. The concepts of finiteness and Dedekind-finiteness are different in FSM, meaning that there exists an infinite invariant set (particularly the set A of atoms, the power set of A, or the set of all bijections of A onto itself) which has no finitely supported injection onto any of its finitely supported proper subsets.

2. There exists an infinite invariant set X (particularly the set A of atoms) such that every finitely supported surjective function from X onto itself is also one-to-one.

3. Let A be the infinite set of atoms in FSM. A finitely supported function $f : A \to A$ is injective if and only if it is surjective.

4. Let $(X, \cdot)$ be an invariant set. Then there exists a finitely supported injection of X onto one of its finitely supported proper subsets if and only if there exists a finitely supported injective mapping $f : \mathbb{N} \to X$.

References

[1] A. Alexandru, G. Ciobanu. *Finitely Supported Mathematics: An Introduction.* Springer, 2016. `doi:10.1007/978-3-319-42282-4_1`.

[2] A. Alexandru, G. Ciobanu. Abstract Interpretations in the Framework of Invariant Sets. *Fundamenta Informaticae* 144(1), 1–22, 2016. `doi:10.3233/fi-2016-1321`.

[3] A. Alexandru, G. Ciobanu. On Logical Notions in the Fraenkel-Mostowski Cumulative Universe. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie* 60/108(2), 113–125, 2017.

[4] A. Alexandru, G. Ciobanu. Fuzzy Sets within Finitely Supported Mathematics. *Fuzzy Sets and Systems* 339, 119–133, 2018. `doi:10.1016/j.fss.2017.08.011`.

[5] J. Barwise. *Admissible Sets and Structures: An Approach to Definability Theory.* Perspectives in Mathematical Logic 7, Springer, 1975.

[6] M. Bojanczyk, B. Klin, S. Lasota. Automata with Group Actions. In: *26th Symposium on Logic in Computer Science*, pages 355–364, IEEE Computer Society Press, 2011. `doi:10.1109/lics.2011.48`.

[7] N. Brunner. Amorphe Potenzen Kompacter Räume. *Archiv für mathematische Logik und Grundlagenforschung* 24, 119–135, 1984. `doi:10.1007/bf02007144`.

[8] A. Fraenkel. Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre. *Mathematische Annalen* 86, 230–237, 1922. `doi:10.1007/bf01457986`.

[9] M.J. Gabbay, A.M. Pitts. A New Approach to Abstract Syntax with Variable Binding. *Formal Aspects of Computing* 13, 341–363, 2001. `doi:10.1007/s001650200016`.

[10] R. Gandy. Church's Thesis and Principles for Mechanisms. In: J. Barwise, H.J. Keisler, K. Kunen, editors, *The Kleene Symposium*, pages 123–148, North Holland, 1980. `doi:10.1016/s0049-237x(08)71257-6`.

[11] L. Halbeisen. *Combinatorial Set Theory, with a Gentle Introduction to Forcing.* Springer, 2011.

[12] H. Herrlich. *Axiom of Choice.* Lecture Notes in Mathematics, Springer, 2006. `doi:10.1007/11601562`.

[13] P. Howard, J.E. Rubin. *Consequences of the Axiom of Choice.* Mathematical Surveys and Monographs, vol.59, American Mathematical Society, 1998. `doi:10.1090/surv/059`.

[14] T.J. Jech. *The Axiom of Choice.* Studies in Logic and the Foundations of Mathematics, North-Holland, 1973.

[15] F. Klein. Vergleichende Betrachtungen Über Neuere Geometrische Forschungen (A Comparative Review of Recent Researches in Geometry). *Mathematische Annalen* 43(1), 63–100, 1893. `doi:10.1007/bf01446615`.

[16] A. Levy. The Independence of Various Definitions of Finite. *Fundamenta Mathematicae* 46(1), 1–13, 1958. `doi:10.4064/fm-46-1-1-13`.

[17] A. Lindenbaum, A. Mostowski. Über die Unabhangigkeit des Auswahlsaxioms und Einiger Seiner Folgerungen. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie* 31, 27–32, 1938.

[18] D. Petrişan. *Investigations Into Algebra and Topology Over Nominal Sets*. Ph.D. Thesis, University of Leicester, 2011.

[19] A.M. Pitts. *Nominal Sets Names and Symmetry in Computer Science*. Cambridge University Press, 2013.

[20] M.R. Shinwell. *The Fresh Approach: Functional Programming with Names and Binders*. Ph.D. Thesis, University of Cambridge, 2005.

[21] A. Tarski. What are Logical Notions?. *History and Philosophy of Logic* 7(2), 143–154, 1986. `doi:10.1080/01445348608837096`.