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Dynamic Chromatic Number of Bipartite Graphs

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Abstract

A *dynamic coloring* of a graph G is a proper vertex coloring such that for every vertex $v \in V(G)$ of degree at least 2, the neighbors of v receive at least 2 colors. The smallest integer k such that G has a dynamic coloring with k colors, is called the *dynamic chromatic number* of G and denoted by $\chi_2(G)$. Montgomery conjectured that for every r-regular graph G, $\chi_2(G) - \chi(G) \leq 2$ [19]. Finding an optimal upper bound for $\chi_2(G) - \chi(G)$ seems to be an intriguing problem. We show that there is a constant d such that every bipartite graph G with $\delta(G) \geq d$, has $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil$. It was shown that $\chi_2(G) - \chi(G) \leq \alpha'(G) + k^*$ [2]. Also, $\chi_2(G) - \chi(G) \leq \alpha(G) + k^*$ [1]. We prove that if G is a simple graph with $\delta(G) > 2$, then $\chi_2(G) - \chi(G) \leq \frac{\alpha'(G)+\omega(G)}{2} + k^*$. Among other results, we prove that for a given bipartite graph $G = [X, Y]$, determining whether G has a dynamic 4-coloring $\ell : V(G) \to \{a, b, c, d\}$ such that a, b are used for part X and c, d are used for part Y is **NP**-complete.

Keywords: Dynamic chromatic number, chromatic number, bipartite graph, computational complexity.

1 Introduction

Throughout this paper all graphs are finite and simple and we follow [23] for terminology and notation are not defined here. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. We denote the

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maximum degree and minimum degree of G by $\Delta(G)$ and $\delta(G)$, respectively. Also, for every $v \in V(G)$ and $X \subseteq V(G)$, $d(v)$, $N(v)$ and $N(X)$ denote the degree of v, the neighbor set of v and the set of vertices of G which has a neighbor in X, respectively. For a natural number k, a graph G is called a k-regular graph if $d(v) = k$, for each $v \in V(G)$.

1.1 Dynamic Coloring of Graphs

A *proper vertex coloring* of a graph G is a function $c : V(G) \longrightarrow L$, such that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A *proper vertex k-coloring* is a proper vertex coloring with $|L| = k$. Let c be a proper vertex coloring of G. Then $c(v)$ and $c(N(v))$ denote the color of v and the set of colors appeared in the neighbors of the vertex v in G, respectively. The smallest integer k such that G has a proper vertex k-coloring is called the *chromatic number* of G and denoted by $\chi(G)$. A proper vertex k-coloring of a graph G is called *dynamic* if for every vertex v with degree at least 2, the neighbors of v receive at least two different colors. The smallest integer k such that G has a dynamic coloring with k colors, is called the *dynamic chromatic number* of G and denoted by $\chi_2(G)$.

During the recent years dynamic coloring of graphs also known as conditional $(k, 2)$-coloring [15, 17, 18] and 2-hued chromatic number of graphs [2]. Recently, the dynamic coloring of graphs has been studied extensively by several authors, for instance see [3, 5, 6, 7, 8, 10, 13, 14].

Remark 1 *For every two integers α, β, where $\alpha \geq \beta \geq 2$, there is a graph $\mathcal{G}_{\alpha,\beta}$ such that $\delta(\mathcal{G}_{\alpha,\beta}) = \beta$, $\Delta(\mathcal{G}_{\alpha,\beta}) = \alpha+1$ and $\chi_2(\mathcal{G}_{\alpha,\beta})-\chi(\mathcal{G}_{\alpha,\beta}) = \lceil \frac{\Delta(\mathcal{G}_{\alpha,\beta})}{\delta(\mathcal{G}_{\alpha,\beta})} \rceil$. To see this for given two numbers α, β where $\alpha \geq \beta \geq 2$, consider the graph $\mathcal{G}_{\alpha,\beta}$ by the following definition: Suppose that $\alpha = \beta c+d$ such that $0 \leq d < \beta$.*

$V(\mathcal{G}_{\alpha,\beta}) = \{v_{i,j} | 1 \leq i \leq \beta, 1 \leq j \leq c+1\}$
$\cup \{v_{i,c+2} | 1 \leq i \leq d\} \cup \{w_k | 1 \leq k \leq c+1\}$,
$E(\mathcal{G}_{\alpha,\beta}) = \{v_{i_1,j_1}v_{i_2,j_2} | j_1 \neq j_2, \{j_1, j_2\} \neq \{c+1, c+2\}\} \cup \{v_{i,j}w_k | j = k\}$.

We have $\chi(\mathcal{G}_{\alpha,\beta}) = c+1$, $\chi_2(\mathcal{G}_{\alpha,\beta}) = 2(c+1)$. It is easy to see that $\delta(\mathcal{G}_{\alpha,\beta}) = \beta$, $\Delta(\mathcal{G}_{\alpha,\beta}) = \alpha+1$ and $\chi_2(\mathcal{G}_{\alpha,\beta}) - \chi(\mathcal{G}_{\alpha,\beta}) = \lceil \frac{\Delta(\mathcal{G}_{\alpha,\beta})}{\delta(\mathcal{G}_{\alpha,\beta})} \rceil$.

It was shown that the difference between the dynamic chromatic number and the chromatic number of a graph can be arbitrarily large [19]. It seems that if the maximum degree is not too large relative to the minimum degree, then $\chi_2(G) - \chi(G)$ is small. Montgomery in [19] proposed the following conjecture.

Conjecture 1 *[Montgomery [19]] For every r-regular graph G, $\chi_2(G) - \chi(G) \leq 2$.*

We say that a set of vertices is *independent* if there is no edge between those vertices. The *independence number*, $\alpha(G)$, of a graph G is the size of a largest independent set of the graph G. There are a lot of efforts to settle Montgomery's conjecture. Alishahi proved that for every k-regular graph G we have $\chi_2(G) - \chi(G) \leq 14.06 \ln k + 1$ [7]. Also, Ahadi et al. proved that if G is a regular graph, then $\chi_2(G) - \chi(G) \leq 2\lfloor \log_2(\alpha(G)) \rfloor + 3$ [2]. Afterwards, it was shown in [8] that for every k-regular graph G with no induced C_4, $\chi_2(G) - \chi(G) \leq 2\lfloor 4 \ln k + 1 \rfloor$. Recently, Taherkhani proved that if G is an k-regular graph, then $\chi_2(G) - \chi(G) \leq \lceil 5.437 \log k + 2.721 \rceil$ [21]. As a generalization of Conjecture 1, Akbari et al. conjectured that for any graph G, $\chi_2(G) - \chi(G) \leq \lceil \frac{\Delta(G)}{\delta(G)} \rceil + 1$ [1]. Alishahi, in [8] gave a negative answer to this conjecture. Recently, Taherkhani in [21] proved that $\chi_2(G) - \chi(G) \leq \mathcal{O}(\frac{\Delta(G)}{\delta(G)} \log(\Delta))$. Here we prove the following:

Theorem 1 *We have the following:*
(i) There is a constant d such that every bipartite graph G with $\delta(G) \geq d$, has $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil$.
(ii) Let $G = [X, Y]$ be a bipartite graph with $\delta(G) \geq 2$, we have $\chi_2(G) - \chi(G) \leq \lceil \dfrac{|V(G)|}{2^{\delta(G)-1}(\delta(G) - 1)} \rceil + 2$.

Let c be a proper vertex coloring for a graph G, then $B_c = \{v \in V(G) \mid d(v) \geq 2,\ |c(N(v))| = 1\}$, also every vertex in B_c is called a *bad vertex*. For every graph G define $k^*(G) = \begin{cases} 2, & \text{if } \chi(G) = 2 \\ 1, & \text{if } \chi(G) \in \{3, 4, 5\} \\ 0, & \text{otherwise.} \end{cases}$

Also, we say that the dynamic property holds for a vertex v, if $|c(N(v))| \geq 2$. A *clique* in a graph $G = (V, E)$ is a subset of its vertices such that every two vertices in the subset are connected by an edge. The *clique number*, $\omega(G)$, of a graph G is the size of a largest clique of the graph G. The maximum number of edges in a matching of a graph G is called the *matching number* of G and denoted $\alpha'(G)$. Finding an optimal upper bound for

$\chi_2(G) - \chi(G)$ seems to be an intriguing problem. It was shown in [2] that $\chi_2(G) - \chi(G) \leq \alpha'(G) + k^*$. Also, $\chi_2(G) - \chi(G) \leq \alpha(G) + k^*$ [1]. We prove the following:

Theorem 2 *If G is a simple graph with $\delta(G) > 2$, then $\chi_2(G) - \chi(G) \leq \frac{\alpha'(G)+\omega(G)}{2} + k^*$.*

In the following remark we show that for simple graph with $\delta(G) = 2$, Theorem 2 is not true.

Remark 2 *For any natural number n, there is a bipartite graph G'_n with $\delta(G'_n) = 2$, such that $\alpha'(G'_n) = n$, $\omega(G'_n) = 2$ and $\chi_2(G'_n) \geq n$. To see this, consider the following graph:*
$V(G'_n) = \{v_1, \ldots, v_n\} \cup \{u_{i,j} | 1 \leq i < j \leq n\}$,
$E(G'_n) = \{v_i u_{i,j}, v_j u_{i,j} | 1 \leq i < j \leq n\}$.
It is easy to see that $\alpha'(G'_n) = n$, $\omega(G'_n) = 2$ and $\chi_2(G'_n) \geq n$.

A *hypergraph* H is a pair (X, Y), where X is the set of vertices and Y is a set of non-empty subsets of X, called edges. The k-coloring of H is a coloring $f : X \to \{1, 2, \ldots, k\}$ such that, for every edge e with $| e |> 1$, there exist $v, u \in X$ such that $f(u) \neq f(v)$. A hypergraph H is bipartite or 2-colorable, if its vertex set can be partitioned into two sets such that every hyperedge intersects both partite sets. The hypergraph H is r-regular if every vertex has degree r in H. A k-edge in H is an edge of size k in H. The hypergraph H is said to be k-uniform if every edge of H is a k-edge. It was shown by Thomassen in [22] that, for any k-uniform and k-regular hypergraph H, if $k \geq 4$, then H is 2-colorable. Therefore, every k-regular bipartite graph $G = [X, Y]$, with $k \geq 4$, has a dynamic 4-coloring $\ell : V(G) \to \{a, b, c, d\}$ such that a, b are used for part X and c, d are used for part Y.

It was shown in [6] that, there are 3-regular bipartite graph $G = [X, Y]$, such that G does not have any dynamic 4-coloring $\ell : V(G) \to \{a, b, c, d\}$ such that a, b are used for part X and c, d are used for part Y. For instance, consider the Fano Plane. The Fano Plane is a hypergraph with seven vertices $\mathbb{Z}_7$ and seven edges $\{\{i, i+1, i+3\} : 1 \leq i \leq 7\}$. Fano plane is not 2-colorable.

A graph G is a $(d, d+s)$-graph if the degree of every vertex of G lies in the interval $[d, d+s]$. A $(d, d+1)$-graph is said to be semiregular.

Theorem 3 *We have the following:*
(i) For a given (2,4)-bipartite graph $H = [L, R]$, determining whether H has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R is **NP***-complete.*
(ii) For a given planar bipartite graph $H = [L, R]$, there is a polynomial time algorithm to determining whether H has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R.

2 Proof of Theorem 1

(i) First, consider the following useful theorem from [16].

Theorem 4 [16] *For a connected graph G if $\Delta(G) \leq 3$, then $\chi_2(G) \leq 4$ unless $G = C_5$, in which case $\chi_2(C_5) = 5$ and if $\Delta(G) \geq 4$, then $\chi_2(G) \leq \Delta(G) + 1$.*

We will use the probabilistic method to prove Theorem 1. The following tool of the probabilistic method will be used (see for instance [9]).

Lemma A [The Lovász Local Lemma [9]**]** *Let $A_1, \ldots, A_n$ be a set of random events such that for each i, $Pr(A_i) \leq p$ and A_i is mutually independent of the set of all but at most d other events. If $4pd \leq 1$, then with positive probability, none of the events occur.*

By Theorem 4, if G is a graph and $1 \leq \delta(G) \leq 2$ or $\Delta(G) \leq 3$, then the theorem is true. So suppose that $G = [X, Y]$ is a bipartite graph with $\delta(G) \geq 2$ and $\Delta(G) > 3$. First, suppose that $\delta(G) \neq \Delta(G)$. We show that we can color the vertices of Part X with $\lceil (4\Delta^2)^{\frac{1}{\delta-1}} \rceil$ colors such that each vertex in Part Y sees at least two colors in its neighbors. To see this, color every vertex of Part X, randomly and independently by one of the colors $\{1, \ldots, \lceil (4\Delta^2)^{\frac{1}{\delta-1}} \rceil\}$, with the same probability. For each vertex v of Part Y, let E_v be the event that all of the neighbors of v have the same color. We have $P(E_v) \leq (\frac{1}{k})^{\delta-1}$. Note that E_v is dependent to $N[v] \cup N[N[v]]$, so depends to at most Δ^2 events. We have $4pd \leq 4\frac{1}{4\Delta^2}\Delta^2 \leq 1$, so by the Local Lemma there is a coloring by $\lceil (4\Delta^2)^{\frac{1}{\delta-1}} \rceil$ colors with a positive probability. By a similar argument, we can color the vertices of Part Y with $\lceil (4\Delta^2)^{\frac{1}{\delta-1}} \rceil$ new colors such that each vertex in Part Y sees at least two colors in its neighbors. Now, three cases can be considered: First suppose that $\frac{\Delta}{\delta} \geq \delta^{\frac{3}{\delta-4}}$. We have:

$$\Delta \geq \delta^{\frac{\delta-1}{\delta-4}},$$
$$\Delta^{\delta-4} \geq \delta^{\delta-1},$$
$$(\tfrac{\Delta}{\delta})^{\delta-1} \geq \Delta^3,$$

Since $\Delta(G) \geq 4$, we have:

$$(\tfrac{\Delta}{\delta})^{\delta-1} \geq 4\Delta^2,$$
$$\tfrac{\Delta}{\delta} \geq (4\Delta^2)^{\frac{1}{\delta-1}}.$$

Hence, we can color each part of the graph G with $\lceil (4\Delta^2)^{\frac{1}{\delta-1}} \rceil$ colors such that each vertex in each part sees at least two colors in its neighbors. Therefore, $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil$.

Next, suppose that $\frac{\Delta}{\delta} \leq \delta^{\frac{3}{\delta-4}}$. There is a constant d, (for example we can assume that $d = 100$) such that if $\delta(G) \geq d$, we have:

$$(4\Delta^2)^{\frac{1}{\delta-1}} \leq (4(\delta)^{\frac{2\delta-2}{\delta-4}})^{\frac{1}{\delta-1}} \leq (4(d)^{\frac{2d-2}{d-4}})^{\frac{1}{d-1}} \leq 3.$$

Note that for $d = 100$, we have $(4(d)^{\frac{2d-2}{d-4}})^{\frac{1}{d-1}} = 1.11$. Since $\delta \neq \Delta$, we have $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil$. Finally, suppose that the graph G is a k-regular bipartite graph with $k \geq 4$. It was shown in [7] that for any k-regular graph G, $\chi_2(G) \leq 2\chi(G)$. Thus, for any k-regular bipartite graph G, $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil$. This completes the proof.

(ii) Let $G = [X, Y]$ be a bipartite graph with $\delta(G) \geq 2$. Let $A \subseteq X$ be a random subset given by $Pr(x \in A) = \frac{1}{2}$, these choices are mutually independent. Set $\overline{A} = X \setminus A$. Given the set A, let $B \subseteq Y$ be the set of vertices having no neighbor in A or having no neighbor in $\overline{A}$. For every vertex $u \in B$, choose one of its neighbors and put it in the set D. If we color the set of vertices A with color 1, also, color the set of vertices $\overline{A}$ with color 2, and partition D into sets of cardinality at most $\delta(G) - 1$, and recolor the vertices of each part with a new color, we obtain a coloring for X from the colors $\{1, \ldots, \lceil \frac{|B|}{\delta(G)-1} \rceil + 2\}$ such that the dynamic property holds for every vertex of Y. Each vertex appears in B with probability at most $\dfrac{1}{2^{\delta(G)-1}}$, thus the expected size of B is $E(B) \leq \dfrac{|Y|}{2^{\delta(G)-1}}$. Similarly, color the vertices of Y with new colors such that the dynamic property holds for every vertex of X. By attention to the value of the expectation, we have $\chi_2(G) \leq \lceil \dfrac{|V(G)|}{2^{\delta(G)-1}(\delta(G)-1)} \rceil + 4$. This completes the proof.

3 Proof of Theorem 2

Let G be a simple graph. Without loss of generality, we can assume that the graph is connected, otherwise we apply our argument for each of its components. In [1] it was proved that for every graph G, there exists a vertex coloring with at most $\chi(G) + k^*$ colors such that the set of bad vertices is independent.

Theorem A [1] *Let G be a graph. Then there exists a vertex $(\chi(G) + k^*)$-coloring of G such that the set of bad vertices of G is independent.*

Let G be a simple graph. By Theorem A, suppose that c is a vertex $(\chi(G) + k^*)$-coloring of G such that B_c is an independent set. Let $M = \{x_1y_1, \ldots, x_{\alpha'}y_{\alpha'}\}$ be a maximum matching of G. Define:

$$X = \{x_1, x_2, \ldots, x_{\alpha'}\} \text{ and } Y = \{y_1, y_2, \ldots, y_{\alpha'}\}$$

Since B_c is an independent set, therefore, there is no index i such that $x_i, y_i \in B_c$. Thus, without loss of generality, suppose that $B_c \cap Y = \emptyset$ (Fact 1).

Since M is a maximum matching, therefore, every vertex $v \in B_c \setminus X$ has a neighbor in $X \cup Y$ (otherwise we find a matching of size $\alpha' + 1$). Also, there is no index i such that $x_i, y_i \in N(B_c \setminus X)$ (otherwise we find a matching of size $\alpha' + 1$). Thus, without loss of generality, suppose that every vertex $v \in B_c \setminus X$ has a neighbor in Y and does not have any neighbor in X (Fact 2).

By Fact 1 and Fact 2, if we recolor the vertices of Y, then the dynamic property holds for every vertex of G. Without loss of generality, consider the following partition

$$\{\{y_1, y_2\}, \{y_3, y_4\}, \ldots, \{v_{2t-1}, v_{2t}\}, \{v_{2t+1}, \ldots, v_l\}\}$$

for the vertices of Y such that for $1 \leq i \leq t$, $v_{2i-1}v_{2i} \notin E(G)$ and $\{v_{2t+1}, \ldots, v_l\}$ is a clique.

For every i, $1 \leq i \leq t$, recolor the vertices v_{2i-1} and v_{2i} with color $\chi(G) + k^* + i$. Next, recolor the vertices $v_{2t+1}, \ldots, v_l$ with different new colors. call the resulting coloring c'. Since $\delta(G) > 2$ and by Fact 1 and Fact 2, c' is a dynamic coloring for G with $\frac{\alpha'(G)+\omega(G)}{2} + k^*$ colors. This completes the proof.

4 Proof of Theorem 3

(i) It was shown that the following problem is **NP**-complete [12].

Cubic Monotone Not-All-Equal (2,3)-Sat.

Instance: Set V of variables, collection C of clauses over V such that each clause $c \in C$ has $|c| \in \{2,3\}$, every variable appears in exactly three clauses and there is no negation in the formula.

Question: Is there a truth assignment for V such that each clause in C has at least one true literal and at least one false literal?

Consider an instance Φ of *Cubic Monotone Not-All-Equal (2,3)-Sat.* Let V be variables, and C collection of clauses over V. We construct a (2,4)-bipartite graph $H = [L, R]$ such that H has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R, if and only if Φ has a Not-All-Equal truth assignment. Our construction consists of two steps.

Step 1.

First, we construct a bipartite graph $G = [X, Y]$. For every variable $v \in V$ put a vertex v in X and for every clause $c \in C$ put a vertex c in Y. Also, for every clause c and variable x, if x appears in c then join c to x in G.

Let $\mathcal{A} = \{A_i : i \in I\}$ be a finite family of (not necessarily distinct) subsets of a finite set $\mathcal{U}$. A *system of distinct representatives (SDR)* for the family $\mathcal{A}$ is a set $\{a_i : i \in I\}$ of distinct elements of $\mathcal{U}$ such that $a_i \in A_i$ for all $i \in I$. Hall's Theorem says that $\mathcal{A}$ has a system of distinct representatives if and only if $|\cup_{i\in J} A_i| \geq |J|$ for all subsets J of I (see for instance [23]). Let $\mathcal{U} = \{c : c \in C\}$ and for every variable $v_i \in V$, let $A_i = \{c : cv_i \in E(G)\}$. In Φ each variable appears three times in the formula and each clause contains at most three variables, Therefore, by Hall's Theorem, there exists a system of distinct representatives of $\mathcal{U}$. For each variable v, denote its representative clause by c_v. Note that there is a polynomial-time algorithm which either finds an SDR or shows that one cannot exist by finding a violation of Halls condition **(Property 1).**

Step 2.

Consider two copies of the graph G, and call them $G = [X, Y]$ and $G' = [X', Y']$ with the vertex set $\{v, c | v \in V, c \in C\}$ and $\{v', c' | v \in V, c \in C\}$, respectively. Put $L = X \cup Y'$, $R = X' \cup Y$, and for every variable $v \in V$, connect the vertex v to the vertex v'. Call the resulting graph $H = [L, R]$ (see Figure 1).

First, suppose that the graph H has a dynamic 4-coloring $\ell : V(H) \to$

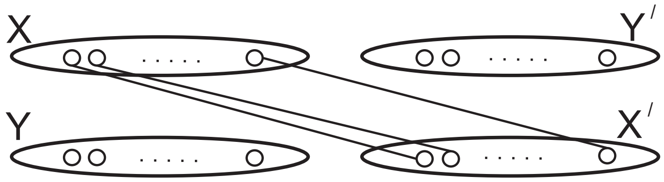

Figure 1: The graph $H = [L, R]$, where $L = X \cup Y'$, $R = X' \cup Y$.

$\{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R. Let $\Gamma : V \to \{true, false\}$ be a function such that $\Gamma(v) = true$ if and only if $\ell(v) = a$. It is easy to see that Γ is a Not-All-Equal satisfying assignment for Φ.

Next, let $\Gamma : V \to \{true, false\}$ be a Not-All-Equal satisfying assignment for Φ. For every variable $v \in V$, put $\ell(v) = a, \ell(v') = c, \ell(c_v) = d, \ell(c_{v'}) = b$ if $\Gamma(v) = true$ and put $\ell(v) = b, \ell(v') = d, \ell(c_v) = c, \ell(c_{v'}) = a$ if $\Gamma(v) = false$. By Property 1, and since Γ is a Not-All-Equal satisfying assignment, therefore the dynamic property holds for every vertex of H. It is easy to extend ℓ into a dynamic coloring for G, such that a, b are used for part L and c, d are used for part R. This completes the proof.

(ii) Moret proved in [20] that *Planar Not-All-Equal 3-Sat* is in **P**. It was shown that the following problem is in **P** (for more information see [4], [11]).

Planar Not-All-Equal Sat Type 2.

Instance: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $| c | \geq 2$ and the following graph obtained from sat is planar. The graph has one vertex for each variable, one vertex for each clause and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause.

Question: Is there a Not-All-Equal truth assignment for X?

For every vertex $v \in L$, consider a variable v in Φ and for every vertex $u \in R$ put a clause $(\bigvee_{v \sim u} v)$ in Φ. Now, determine whether the formula Φ has a Not-All-Equal truth assignment. If the formula Φ does not have a Not-All-Equal truth assignment, then the graph H does not have any dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R.

If the formula Φ has a Not-All-Equal truth assignment, we perform a similar procedure for the graph H. For every vertex $v \in R$, consider a variable v in Φ' and for every vertex $u \in L$ put a clause $(\bigvee_{v\sim u} v)$ in Φ'. Now, determine whether Φ' has a Not-All-Equal truth assignment. If Φ' does not have a Not-All-Equal truth assignment, then H does not have any dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R.

If Φ and Φ' have Not-All-Equal truth assignments, it is easy to find a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R.

5 Concluding Remarks and Future Work

• Finding an optimal upper bound for $\chi_2(G) - \chi(G)$ seems to be an intriguing problem for future work. In this work, we proved that there is a constant d such that every bipartite graph G with $\delta(G) \geq d$, has $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil$. So, for every bipartite graph G, we have $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil + \mathcal{O}(1)$. Finding the minimum number c such that every bipartite graph G, has $\chi_2(G) - \chi(G) \leq 2\lceil \frac{\Delta(G)}{\delta(G)} \rceil + c$ is an interesting work.

• In this work we proved that for a given (2,4)-bipartite graph $H = [L, R]$, determining whether H has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R is **NP**-complete. On the other hand, we show that for each $r > 3$, every r-regular bipartite graph $H = [L, R]$ has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R. The following problems remains unsolved.

Problem 1 Determine the computational complexity of deciding whether a given 3-regular bipartite graph $H = [L, R]$ has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R.

We conclude this section with the following conjecture:

Conjecture 2 *for a given (2,3)-bipartite graph $H = [L, R]$, determining whether H has a dynamic 4-coloring $\ell : V(H) \to \{a, b, c, d\}$ such that a, b are used for part L and c, d are used for part R is* **NP**-*complete.*

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