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Locality and Applications to Subsumption Testing in $\mathcal{EL}$ and Some of its Extensions

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Abstract

In this paper we show that subsumption problems in the description logics $\mathcal{EL}$ and $\mathcal{EL}^+$ can be expressed as uniform word problems in classes of semilattices with monotone operators. We use possibilities of efficient local reasoning in such classes of algebras, to obtain uniform PTIME decision procedures for TBox and CBox subsumption in $\mathcal{EL}$ and $\mathcal{EL}^+$. These locality considerations allow us to present a new family of (possibly many-sorted) logics which extend $\mathcal{EL}$ and $\mathcal{EL}^+$ with n-ary roles and/or numerical domains.

Keywords: description logics, deduction, hierarchical reasoning

1 Introduction

Description logics are logics for knowledge representation used in databases and ontologies. They provide a logical basis for modeling and reasoning about objects, classes (or concepts), and relationships (or links, or roles) between them. Recently, tractable description logics such as $\mathcal{EL}$ [2] have attracted much interest. Although they have relatively restricted expressivity, this expressivity is sufficient for formalizing the type of knowledge used in widely used ontologies such as the medical ontology SNOMED [26, 27]. Several papers were dedicated to studying the properties of $\mathcal{EL}$ and of its extensions (e.g. $\mathcal{EL}^+$ [4]), especially to understanding the limits of tractability in extensions of $\mathcal{EL}$. Undecidability results in extensions of $\mathcal{EL}$ are obtained in [1] using a reduction to the word problem for semi-Thue systems.

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In this paper we show that the subsumption problem in $\mathcal{EL}$ and $\mathcal{EL}^+$ can be expressed as satisfiability problems for ground clauses w.r.t. *local and stably local (extensions of) theories*, for which methods for efficient (PTIME) checking of satisfiability of ground clauses exist. The results on local theories allow us to extend the tractability results to some extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$ with n-ary roles and/or numerical domains. The results were first presented in [24]. This paper extends [24] by giving more details on stably local theory extensions and providing additional examples.

Structure of the paper. In Section 2 we present generalities on description logic and introduce the description logics $\mathcal{EL}$ and $\mathcal{EL}^+$. In Section 3 we show that CBox subsumption can be expressed as a uniform word problem in the class of semilattices with monotone operators satisfying certain composition axioms. In Section 4 we present general definitions and results on local and stably local equational theories and in Section 5 we show that the algebraic models of $\mathcal{EL}$ and $\mathcal{EL}^+$ have local resp. stably local presentations, thus providing an alternative proof of the fact that TBox subsumption in $\mathcal{EL}$ and $CBox$ subsumption in $\mathcal{EL}^+$ are decidable in PTIME. Locality results for more general classes of semilattice with operators are used in Section 6 for defining tractable extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$.

2 Description Logics: Generalities

The central notions in description logics are concepts and roles. In any description logic a set N_C of *concept names* and a set N_R of *roles* is assumed to be given. Complex concepts are defined starting with the concept names in N_C, with the help of a set of *concept constructors*. The available constructors determine the expressive power of a description logic.

The semantics of description logics is defined in terms of interpretations $\mathcal{I} = (D^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $D^{\mathcal{I}}$ is a non-empty set, and the function $\cdot^{\mathcal{I}}$ maps each concept name $C \in N_C$ to a set $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ and each role name $r \in N_R$ to a binary relation $r^{\mathcal{I}} \subseteq D^{\mathcal{I}} \times D^{\mathcal{I}}$.

Table 1 shows the constructor names used in the description logic $\mathcal{ALC}$ and their semantics. The extension of $\cdot^{\mathcal{I}}$ to concept descriptions is inductively defined using the semantics of the constructors.

Terminology. A *terminology* (or TBox, for short) is a finite set consisting of *primitive concept definitions* of the form $C \equiv D$, where C is a concept

Table 1: Constructors and their semantics

Constructor name	Syntax	Semantics
negation	$\neg C$	$D^{\mathcal{I}} \backslash C^{\mathcal{I}}$
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
disjunction	$C_1 \sqcup C_2$	$C_1^{\mathcal{I}} \cup C_2^{\mathcal{I}}$
existential restriction	$\exists r.C$	$\{x \mid \exists y \ ((x,y) \in r^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}})\}$
universal restriction	$\forall r.C$	$\{x \mid \forall y \ (\text{if } (x,y) \in r^{\mathcal{I}} \text{ then } y \in C^{\mathcal{I}})\}$

name and D a concept description; and *general concept inclusions* (GCI) of the form $C \sqsubseteq D$, where C and D are concept descriptions.

Interpretations. An interpretation $\mathcal{I}$ is a model of a TBox $\mathcal{T}$ if it satisfies:

- all concept definitions in $\mathcal{T}$, i.e. $C^{\mathcal{I}}{=}D^{\mathcal{I}}$ for all definitions $C{\equiv}D \in \mathcal{T}$;
- all general concept inclusions in $\mathcal{T}$, i.e. $C^{\mathcal{I}}{\subseteq}D^{\mathcal{I}}$ for every $C{\sqsubseteq}D \in \mathcal{T}$.

Since definitions can be expressed as double inclusions, in what follows we will only refer to TBoxes consisting of general concept inclusions (GCI) only.

Definition 1 *Let $\mathcal{T}$ be a TBox, and C_1, C_2 two concept descriptions. C_1 is subsumed by C_2 w.r.t. $\mathcal{T}$ (for short, $C_1 \sqsubseteq_{\mathcal{T}} C_2$) if and only if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{T}$.*

Relationships between concepts and roles are described using CBoxes.

Constraint box. A CBox consists of a terminology $\mathcal{T}$ and a set RI of role inclusions of the form $r_1 \circ \cdots \circ r_n \sqsubseteq s$. (Since any terminology can be expressed as a set of general concept inclusions, in what follows we will view CBoxes as unions $GCI \cup RI$ of a set GCI of general concept inclusions and a set RI of role inclusions of the form $r_1 \circ \cdots \circ r_n \sqsubseteq s$.)

Interpretation. An interpretation $\mathcal{I}$ is a model of the CBox $\mathcal{C} = GCI \cup RI$ if it is a model of GCI and satisfies all role inclusions in $\mathcal{C}$, i.e. $r_1^{\mathcal{I}} \circ \cdots \circ r_n^{\mathcal{I}} \subseteq s^{\mathcal{I}}$ for all $r_1 \circ \cdots \circ r_n \subseteq s \in RI$.

Definition 2 *If $\mathcal{C}$ is a CBox, and C_1, C_2 are concept descriptions then $C_1 \sqsubseteq_{\mathcal{C}} C_2$ if and only if $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for every model $\mathcal{I}$ of $\mathcal{C}$.*

2.1 The Description Logics $\mathcal{EL}$ and $\mathcal{EL}^+$

By restricting the type of allowed concept constructors, less expressive but tractable description logics can be defined. If we only allow intersection and existential restriction as concept constructors, we obtain the description logic $\mathcal{EL}$ [2], a logic used in terminological reasoning in medicine [26, 27]. In [4], the extension $\mathcal{EL}^+$ of $\mathcal{EL}$ with role inclusion axioms is studied. It was shown that subsumption w.r.t. CBoxes in $\mathcal{EL}^+$ can be reduced in linear time to subsumption w.r.t. *normalized* CBoxes, in which all GCIs have one of the forms:

$$\begin{array}{rl} C \sqsubseteq D, & C \sqsubseteq \exists r.D, \\ C_1 \sqcap C_2 \sqsubseteq D, & \exists r.C \sqsubseteq D, \end{array}$$

where C, C_1, C_2, D are concept names, and all role inclusions are of the form

$$r \sqsubseteq s \quad \text{or} \quad r_1 \circ r_2 \sqsubseteq r.$$

Therefore, in what follows, we consider w.l.o.g. that CBoxes only contain role inclusions of the form $r \sqsubseteq s$ and $r_1 \circ r_2 \sqsubseteq r$.

3 Algebraic Semantics for $\mathcal{EL}$ and $\mathcal{EL}^+$

We show that CBox subsumption for $\mathcal{EL}$ and $\mathcal{EL}^+$ can be expressed as a uniform word problem for classes of semilattices with monotone operators.

3.1 Algebra: Preliminaries

We assume known notions such as partially-ordered set and order filter/ideal in a partially-ordered set. For further information cf. [8]. A structure $(L, \wedge)$ consisting of a non-empty set L together with a binary operation $\wedge$ is called *semilattice* if $\wedge$ is associative, commutative and idempotent. A structure $(L, \vee, \wedge)$ consisting of a non-empty set L together with two binary operations $\vee$ and $\wedge$ on L is called *lattice* if $\vee$ and $\wedge$ are associative, commutative and idempotent and satisfy the absorption laws. A *distributive lattice* is a lattice that satisfies either of the distributive laws $(D_\wedge)$ or $(D_\vee)$, which are equivalent in a lattice.

$$\begin{array}{lll} (D_\wedge) & \forall x, y, z & x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \\ (D_\vee) & \forall x, y, z & x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z). \end{array}$$

A lattice having both a first and a last element is called *bounded*. A Boolean algebra is a structure $(B, \vee, \wedge, \neg, 0, 1)$, such that $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $\neg$ is a unary operation that satisfies:

$$(\mathsf{Complement}) \qquad \forall x \quad \neg x \vee x = 1 \qquad\qquad \forall x \quad \neg x \wedge x = 0$$

Let $\mathcal{V}$ be a class of algebras. The *universal Horn theory* of $\mathcal{V}$ is the collection of those closed formulae valid in $\mathcal{V}$ which are of the form

$$\forall x_1 \dots \forall x_n (\bigwedge_{i=1}^{n} s_{i1} = s_{i2} \rightarrow t_1 = t_2) \qquad (1)$$

The formula (1) above is valid in $\mathcal{V}$ if for each algebra $\mathcal{A} \in \mathcal{V}$ and each assignment v of values in A to the variables, if $v(s_{i1}) = v(s_{i2})$ for all $i \in \{1, \dots, n\}$ then $v(t_1) = v(t_2)$.[2] The problem of deciding the validity of universal Horn sentences in a class $\mathcal{V}$ of algebras is also called the *uniform word problem* for $\mathcal{V}$. It is known that the uniform word problem is decidable for the classes: SL of semilattices (in PTIME), DL of distributive lattices (co-NP-complete), and Bool of Boolean algebras (co-NP-complete).

3.2 An Algebraic Semantics for Description Logics

A translation of concept descriptions into terms in a signature naturally associated with the set of constructors can be defined as follows. For every role name r, we introduce unary function symbols, $f_{\exists r}$ and $f_{\forall r}$. The renaming is inductively defined by:

- $\overline{C} = C$ for every concept name C;
- $\overline{\neg C} = \neg \overline{C}$; $\quad \overline{C_1 \sqcap C_2} = \overline{C}_1 \wedge \overline{C}_2$, $\quad \overline{C_1 \sqcup C_2} = \overline{C}_1 \vee \overline{C}_2$;
- $\overline{\exists r.C} = f_{\exists r}(\overline{C})$, $\quad \overline{\forall r.C} = f_{\forall r}(\overline{C})$.

Set theoretical semantics. There exists a one-to-one correspondence between interpretations of description logics, $\mathcal{I} = (D, \cdot^{\mathcal{I}})$ and Boolean algebras of sets $(\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\exists r}, f_{\forall r}\}_{r \in N_R})$, together with valuations $v : N_C \rightarrow \mathcal{P}(D)$, where $f_{\exists r}, f_{\forall r}$ are defined, for every $U \subseteq D$, by:

$$\begin{array}{rcl} f_{\exists r}(U) & = & \{x \mid \exists y \ ((x, y) \in r^{\mathcal{I}} \text{ and } y \in U)\} \\ f_{\forall r}(U) & = & \{x \mid \forall y \ (\text{if } (x, y) \in r^{\mathcal{I}} \text{ then } y \in U)\}. \end{array}$$

[2] If A is an algebra and $v : X \rightarrow A$ an assignment, then v extends in a canonical way to a homomorphism $\overline{v}$ from the algebra of terms with variables X to A. For every term t with variables in X we will, for the sake of simplicity, write $v(t)$ instead of $\overline{v}(t)$.

Let $v : N_C \to \mathcal{P}(D)$ with $v(A) = A^{\mathcal{I}}$ for all $A \in N_C$, and let $\overline{v}$ be the (unique) homomorphic extension of v to terms. Let C be a concept description and $\overline{C}$ be its associated term. Then $C^{\mathcal{I}} = \overline{v}(\overline{C})$ (denoted by $\overline{C}^{\mathcal{I}}$).

Boolean algebras with operators. Let BAO_{N_R} be the class of all Boolean algebras with operators $(B, \vee, \wedge, \neg, 0, 1, \{f_{\exists r}, f_{\forall r}\}_{r \in N_R})$, where

- $f_{\exists r}$ is a join hemimorphism, i.e. $f_{\exists r}(x \vee y) = f_{\exists r}(x) \vee f_{\exists r}(y)$, $f_{\exists r}(0) = 0$;
- $f_{\forall r}$ is a meet hemimorphism, i.e. $f_{\forall r}(x \wedge y){=}f_{\forall r}(x) \wedge f_{\forall r}(y)$, $f_{\forall r}(1){=}1$;
- $f_{\forall r}(x) = \neg f_{\exists r}(\neg x)$ for every $x \in B$.

It is known that the TBox subsumption problem for $\mathcal{ALC}$ can be expressed as uniform word problem for Boolean algebras with suitable operators.

Theorem 1 *If $\mathcal{T}$ is an $\mathcal{ALC}$ TBox consisting of general concept inclusions between concept terms formed from concept names $N_C = \{C_1, \dots, C_n\}$, and D_1, D_2 are concept descriptions, the following are equivalent:*

(1) $D_1 \sqsubseteq_{\mathcal{T}} D_2$.

(2) $\mathcal{P}(\mathbf{D}) \models \forall C_1 ... C_n \left(\left(\bigwedge_{C \sqsubseteq D \in \mathcal{T}} \overline{C} \leq \overline{D} \right) \to \overline{D_1} \leq \overline{D_2} \right)$
for all interpretations $\mathcal{I} = (D, \cdot^{\mathcal{I}})$,
where $\mathcal{P}(\mathbf{D}) = (\mathcal{P}(D), \cup, \cap, \neg, \emptyset, D, \{f_{\exists r}, f_{\forall r}\}_{r \in N_R})$.

(3) $\mathsf{BAO}_{N_R} \models \forall C_1 ... C_n \left(\left(\bigwedge_{C \sqsubseteq D \in \mathcal{T}} \overline{C} \leq \overline{D} \right) \to \overline{D_1} \leq \overline{D_2} \right)$.

Proof: The equivalence of (1) and (2) follows from the definition of $D_1 \sqsubseteq_{\mathcal{T}} D_2$. (3) ⇒ (2) is immediate. (2) ⇒ (3) follows from the fact that every algebra in BAO_{N_R} homomorphically embeds into a Boolean algebra of sets. □

3.3 An Algebraic Semantic for $\mathcal{EL}^+$

In [19] we studied the link between TBox subsumption in $\mathcal{EL}$ and uniform word problems in the corresponding classes of semilattices with monotone functions. We now show that these results naturally extend to the description logic $\mathcal{EL}^+$. Consider the following classes of algebras:

- $\mathsf{BAO}^{\exists}_{N_R}$ the class of boolean algebras with operators $(B, \vee, \wedge, \neg, 0, 1, \{f_{\exists r}\}_{r \in N_R})$, such that $f_{\exists r}$ is a join hemimorphism;
- $\mathsf{DLO}^{\exists}_{N_R}$ the class of bounded distributive lattices with operators $(L, \vee, \wedge, 0, 1, \{f_{\exists r}\}_{r \in N_R})$, such that $f_{\exists r}$ is a join hemimorphism;
- $\mathsf{SLO}^{\exists}_{N_R}$ the class of all $\wedge$-semilattices with operators $(S, \wedge, 0, 1, \{f_{\exists R}\}_{R \in N_R})$, such that $f_{\exists R}$ is monotone and $f_{\exists R}(0) = 0$. [3]

Assume given a set RI of axioms of the form $r \sqsubseteq s$ and $r_1 \circ r_2 \sqsubseteq r$, with $r_1, r_2, r, s \in N_R$. We associate with RI the following set of axioms:

$$\begin{array}{rcl} RI_a & = & \{\forall x \;\; (f_{\exists r_1}(f_{\exists r_2}(x)) \leq f_{\exists r}(x) \mid r_1 \circ r_2 \sqsubseteq r \in RI\} \cup \\ & & \{\forall x \;\; f_{\exists r}(x) \leq f_{\exists s}(x) \mid r \sqsubseteq s \in RI\}. \end{array}$$

Let $\mathsf{BAO}^{\exists}_{N_R}(RI)$ (resp.$\mathsf{DLO}^{\exists}_{N_R}(RI)$, $\mathsf{SLO}^{\exists}_{N_R}(RI)$) be the subclass of $\mathsf{BAO}^{\exists}_{N_R}$ (resp. $\mathsf{DLO}^{\exists}_{N_R}$, $\mathsf{SLO}^{\exists}_{N_R}$) consisting of those algebras which satisfy RI_a.

Lemma 1 *Let $\mathcal{I} = (D, \cdot^{\mathcal{I}})$ be a model of an $\mathcal{EL}^+$ CBox, $\mathcal{C} = GCI \cup RI$. Then $(\mathcal{P}(D), \cap, \{f_{\exists r}\}_{r \in N_R}) \in \mathsf{SLO}^{\exists}_{N_R}(RI)$.*

Proof: Clearly, $(\mathcal{P}(D), \cap, \{f_{\exists r}\}_{r \in N_R}) \in \mathsf{SLO}^{\exists}_{N_R}$. Let $r_1, r_2, r \in N_R$ and $U \in \mathcal{P}(D)$.

$$\begin{array}{rcll} f_{\exists r_1}(U) & = & \{x \mid \exists y \in U \text{ s.t. } (x, y) \in r_1^{\mathcal{I}}\} \;\subseteq\; f_{\exists r}(U) & \text{if } r_1 \sqsubseteq r \in RI \\ f_{\exists r_1}(f_{\exists r_2}(U)) & = & \{x \mid \exists y \text{ s.t. } (x, y) \in r_1^{\mathcal{I}} \text{ and } y \in f_{\exists r_2}(U)\} & \\ & = & \{x \mid \exists y \text{ s.t. } (x, y) \in r_1^{\mathcal{I}} \text{ and } \exists z \in U \text{ with } (y, z) \in r_2^{\mathcal{I}}\} & \\ & = & \{x \mid \exists z \in U \text{ s.t. } (x, z) \in (r_1 \circ r_2)^{\mathcal{I}}\} & \\ & \subseteq & f_{\exists r}(U) & \text{if } r_1 \circ r_2 \sqsubseteq r \in RI. \end{array}$$

Theorem 2 *Every $\mathbf{S} \in \mathsf{SLO}^{\exists}_{N_R}(RI)$ embeds into (the bounded semilattice reduct of) a lattice in $\mathsf{DLO}^{\exists}_{N_R}(RI)$. Every lattice in $\mathsf{DLO}^{\exists}_{N_R}(RI)$ embeds into (the bounded lattice reduct of) an algebra in $\mathsf{BAO}^{\exists}_{N_R}(RI)$.*

Proof: Let $\mathbf{S} = (S, \wedge, 0, 1, \{f_S\}_{f \in \Sigma})$ be a semilattice with 0, 1, and with monotone operators in Σ such that $f_S(0) = 0$. Consider the lattice of all

[3] For the sake of simplicity, in what follows we assume that the description logics $\mathcal{EL}$ and $\mathcal{EL}^+$ contain the additional constructors $\bot, \top$, which will be interpreted as 0 and 1. Similar considerations can be used to show that the algebraic semantics for variants of $\mathcal{EL}$ and $\mathcal{EL}^+$ having only $\top$ (or $\bot$) is given by semilattices with 1 (resp. 0).

order-ideals of S, $\mathcal{OI}(\mathbf{S}) = (\mathcal{OI}(\mathbf{S}), \cap, \cup, \{0\}, S, \{\overline{f}_S\}_{f\in\Sigma})$, where join is set union, meet is set intersection, and the additional operators in Σ are defined, for every order ideal U of S, by $\overline{f}_S(U) = {\downarrow} f_S(U)$. Note that

$$\begin{array}{rcl} \overline{f}(\{0\}) & = & \{0\} \\ \overline{f}(U_1 \vee U_2) & = & {\downarrow} f(U_1 \vee U_2) = {\downarrow}(f(U_1) \cup f(U_2)) = {\downarrow} f(U_1) \cup {\downarrow} f(U_2). \end{array}$$

Thus, $\mathcal{OI}(\mathbf{S}) \in \mathsf{DLO}^{\exists}_{N_R}$. [4] Moreover, $\eta : \mathbf{S} \to \mathcal{OI}(\mathbf{S})$ defined by $\eta(x) := {\downarrow} x$ is an injective homomorphism w.r.t. the operations in SLO_{N_R}, i.e. $\eta(f_S(x)) = {\downarrow} f_S(x) = \overline{f}_S({\downarrow} x)$. Let $r_1 \circ r_2 \sqsubseteq r \in RI$, and let $U \in \mathcal{OI}(\mathbf{S})$. Then:

$$\begin{array}{rcl} \overline{f}_{\exists r_1}(U) & = & {\downarrow} f_{\exists r_1}(U), \\ \overline{f}_{\exists r_1}(\overline{f}_{\exists r_2}(U)) & = & \overline{f}_{\exists r_1}({\downarrow} f_{\exists r_2}(U)) = {\downarrow} f_{\exists r_1}(f_{\exists r_2}(U)). \end{array}$$

The second statement is a consequence of Priestley duality for distributive lattices. Let $\mathbf{L} \in \mathsf{DLO}^{\exists}_{N_R}(RI)$. Let $\mathcal{F}_p$ be the set of prime filters of L, and $B(\mathbf{L}) = (\mathcal{P}(\mathcal{F}_p), \cup, \cap, \{\overline{f}_{\exists r}\}_{r \in N_r})$, where for $r \in R$, $\overline{f}_{\exists r}$ is defined by

$$\overline{f}_{\exists r}(U) = \{F \in \mathcal{F}_p \mid \exists G \in U : f_{\exists r}(G) \subseteq F\}.$$

Let $i : \mathbf{L} \to B(\mathbf{L})$ be defined by $i(x) = \{F \in \mathcal{F}_p \mid x \in F\}$. Obviously, i is a lattice homomorphism. We show that $i(f_{\exists r}(x)) = \overline{f}_{\exists r}(i(x))$.

$$\begin{array}{rcl} \overline{f}_{\exists r}(i(x)) & = & \{F \in \mathcal{F}_p \mid \exists G : x \in G \text{ and } f_{\exists r}(G) \subseteq F\} \\ & \subseteq & \{F \in \mathcal{F}_p \mid f_{\exists r}(x) \in F\} = i(f_{\exists r}(x)). \end{array}$$

To prove the converse inclusion, let $F \in i(f_{\exists r}(x))$. Then $F \in \mathcal{F}_p$ and $f_{\exists r}(x) \in F$. Then $x \in G = f_{\exists r}^{-1}(F)$. As F is a prime filter, and $f_{\exists r}$ is a join hemimorphism, $G = f_{\exists r}^{-1}(F)$ is a prime filter with $x \in G$ and $f_{\exists r}(G) \subseteq F$, so $F \in \overline{f}_{\exists r}(i(x))$.

Finally, we show that $B(\mathbf{L})$ satisfies the axioms in RI_a. Let $U \in B(\mathbf{L})$. Then:

$$\begin{array}{rcl} \overline{f}_{\exists r_1}(U) & = & \{F \in \mathcal{F}_p \mid \exists G_1 \in U : f_{\exists r_1}(G_1) \subseteq F\}, \\ \overline{f}_{\exists r_1}(\overline{f}_{\exists r_2}(U)) & = & \{F \in \mathcal{F}_p \mid \exists G_1, \exists G_2 \in U : f_{\exists r_2}(G_2) \subseteq G_1 \text{ and } f_{\exists r_1}(G_1) \subseteq F\} \\ & \subseteq & \{F \in \mathcal{F}_p \mid \exists G_2 \in U : f_{\exists r_1}(f_{\exists r_2}(G_2)) \subseteq F\}. \end{array}$$

[4] A similar construction can be made starting from $\wedge$-semilattice with monotone operators which have only 1 (resp. 0) or neither 0 nor 1.

Assume that $r_1 \sqsubseteq r \in RI$. Then for all x, $f_{\exists r_1}(x) \leq f_{\exists r}(x)$. Let $F \in \overline{f}_{\exists r_1}(U)$. Then $f_{\exists r_1}(G_1) \subseteq F$ for some $G_1 \in U$, so also $f_{\exists r}(G_1) \subseteq F$. Hence, $\overline{f}_{\exists r_1}(U) \subseteq \overline{f}_{\exists r}(U)$. Similarly we can prove that if $r_1 \circ r_2 \sqsubseteq r \in RI$ then $\overline{f}_{\exists r_1}(\overline{f}_{\exists r_2}(U)) \subseteq \overline{f}_{\exists r}(U)$. □

Theorem 3 *If the only concept constructors are intersection and existential restriction, then for all concept descriptions D_1, D_2 and every $\mathcal{EL}^+$ CBox $\mathcal{C}$=GCI∪RI, with concept names $N_C = \{C_1, \dots, C_n\}$ the following are equivalent:*

(1) $D_1 \sqsubseteq_{\mathcal{C}} D_2$.

(2) $\mathsf{SLO}^{\exists}_{N_R}(RI) \models \forall C_1 \dots C_n \left(\left(\bigwedge_{C \sqsubseteq D \in GCI} \overline{C} \leq \overline{D} \right) \rightarrow \overline{D_1} \leq \overline{D_2} \right).$

Proof: We know that $C_1 \sqsubseteq_{\mathcal{C}} C_2$ iff $C_1^{\mathcal{I}} \subseteq C_2^{\mathcal{I}}$ for every model $\mathcal{I}$ of the CBox $\mathcal{C}$.

We first prove that (1) ⇒ (2). Assume that (2) holds. Let $\mathcal{I} = (D, \cdot^{\mathcal{I}})$ be an interpretation that satisfies $\mathcal{C}$. Then $(\mathcal{P}(D), \cap, \emptyset, D, \{f_{\exists r}\}_{r \in N_R}) \in \mathsf{SLO}^{\exists}_{N_R}(RI)$, hence $(\mathcal{P}(D), \cap, \emptyset, D, \{f_{\exists r}\}_{r \in N_R}) \models \left(\bigwedge_{C \sqsubseteq D \in GCI} \overline{C} \leq \overline{D} \right) \rightarrow \overline{D_1} \leq \overline{D_2}$. As $\mathcal{I}$ is a model of GCI, $\overline{C}^{\mathcal{I}} \subseteq \overline{D}^{\mathcal{I}}$ for all $C \sqsubseteq D \in GCI$, so $D_1^{\mathcal{I}} = \overline{D_1}^{\mathcal{I}} \subseteq \overline{D_2}^{\mathcal{I}} = D_2^{\mathcal{I}}$.

To prove (1) ⇒ (2) note that, by Theorem 1, if $D_1 \sqsubseteq_{\mathcal{T}} D_2$ then $\mathsf{BAO}_{N_R} \models \left(\bigwedge_{C \sqsubseteq D \in \mathcal{C}} \overline{C} \leq \overline{D} \right) \rightarrow \overline{D_1} \leq \overline{D_2}$. Let $\mathbf{S} \in \mathsf{SLO}^{\exists}_{N_R}(RI)$. By Lemma 2, $\mathbf{S}$ embeds into an algebra in $\mathsf{BAO}^{\exists}_{N_R}$ which satisfies RI_a. Therefore, $\mathbf{S} \models \left(\bigwedge_{C \sqsubseteq D \in GCI} \overline{C} \leq \overline{D} \right) \rightarrow \overline{C_1} \leq \overline{C_2}$. □

We will show that the word problem for the class of algebras $\mathsf{SLO}^{\exists}_{N_R}(RI)$ is decidable in PTIME. For this we will prove that $\mathsf{SLO}^{\exists}_{N_R}(RI)$ has a "local" presentation. The general locality definitions, as well as methods for recognizing local presentations are given in Section 4. We show how to apply these methods for the class of algebraic models of $\mathcal{EL}$ and $\mathcal{EL}^+$ in Section 5.

4 Local Theories; Local Theory Extensions

First-order theories are sets of formulae (closed under logical consequence), typically the set of all consequences of a set of axioms. Alternatively, we may consider the set of all models of a theory. In this paper we consider theories specified by their sets of axioms. (At places, however, we will refer to a theory, and mean the set of all its models.) Before defining the notion of local theory and local theory extension we will introduce some preliminary notions on partial models of a theory.

4.1 Partial and Total Models

A partial model is a model in which some function symbols may be partial. In this paper the models we consider are partially ordered algebraic structures, i.e. the only predicates are $\leq$ and $=$.

Definition 3 *A* weak Π-embedding *between the partial structures* $A = (\{A_s\}_{s\in S}, \{f_A\}_{f\in\Sigma}, \{P_A\}_{P\in\mathsf{Pred}})$ *and* $B = (\{B_s\}_{s\in S}, \{f_B\}_{f\in\Sigma}, \{P_B\}_{P\in\mathsf{Pred}})$ *is a (many-sorted) family* $i = (i_s)_{s\in S}$ *of total maps* $i_s : A_s \to B_s$ *such that*

- *if* $f_A(a_1, \dots, a_n)$ *is defined then also* $f_B(i_{s_1}(a_1), \dots, i_{s_n}(a_n))$ *is defined and* $i_s(f_A(a_1, \dots, a_n)) = f_B(i_{s_1}(a_1), \dots, i_{s_n}(a_n))$*;*
- *for each* s*,* i_s *is injective and an embedding w.r.t.* Pred *i.e. for every* $P \in \mathsf{Pred}$ *with arity* $s_1 \dots s_n$ *and every* $a_1, \dots, a_n$ *where* $a_i \in A_{s_i}$*,* $P_A(a_1, \dots, a_n)$ *if and only if* $P_B(i_{s_1}(a_1), \dots, i_{s_n}(a_n))$*.*

In this case we say that A weakly embeds *into* B*.*

Definition 4 *Let* A *be a partial structure and* $\beta : X \to A$ *be a valuation.*

- *We say that* $(A, \beta) \models t_1 = t_2$ *iff at least one of the following conditions holds:*
  - *(a)* $\beta(t_1), \beta(t_2)$ *are defined and* $\beta(t_1) = \beta(t_2)$*, or*
  - *(b)* $\beta(t_1)$ *and* $\beta(t_2)$ *are undefined, or*
  - *(c)* $\beta(t_1)$ *is defined,* $t_2{=}f(s_1, \dots, s_n)$ *and* $\beta(s_i)$ *is undefined for some* i*, or*
  - *(d) if* $\beta(t_1)$ *is defined,* $t_2 = f(s_1, \dots, s_n)$ *and* $\beta(s_i)$ *is defined for all* i *then* $\beta(t_2)$ *has to be defined and* $\beta(t_1) = \beta(t_2)$*.*
- $(A, \beta) \models t_1 \leq t_2$ *is defined similarly, replacing "$=$" with "$\leq$" in (a)–(d).*
- *We say that* $(A, \beta) \models t_1 \neq t_2$ *if at least one of the following conditions holds:*

(a') $\beta(t_1), \beta(t_2)$ are defined and $\beta(t_1) \neq \beta(t_2)$, or

(b') $\beta(t_1)$ or $\beta(t_2)$ are undefined.

- (A, β) satisfies a clause C *(notation: $(A, \beta) \models C$) if it satisfies at least one literal in C.*

Definition 5 *A is an* (Evans) partial model *of a set of clauses $\mathcal{K}$ if $(A, \beta) \models C$ for every valuation β and every clause C in $\mathcal{K}$.*

Definition 6 *Let A be a partial structure and $\beta : X \to A$ be a valuation.*

- *We say that $(A, \beta) \models_w (\neg)P(t_1, \dots, t_n)$ ((A, β) weakly satisfies $(\neg)P(t_1, \dots, t_n)$), with $P \in \mathsf{Pred} \cup \{=\}$ if either $\beta(t_i)$ are all defined and $(\neg)P_A(\beta(t_1), \dots, \beta(t_n))$ is true in A, or $\beta(t_i)$ is not defined for some argument t_i of P.*
- (A, β) weakly satisfies a clause C *(notation: $(A, \beta) \models_w C$) if it weakly satisfies satisfies at least one literal in C.*

Definition 7 *We say that A is a* weak partial model *of a set of clauses $\mathcal{K}$ if $(A, \beta) \models_w C$ for every $\beta : X \to A$ and $C \in \mathcal{K}$.*

4.2 Local Theories

The notion of local theory was introduced by Givan and McAllester [11, 12]. They studied sets of Horn clauses $\mathcal{K}$ with the property that, for any ground Horn clause C, $\mathcal{K} \models C$ only if already $\mathcal{K}[C] \models C$ (where $\mathcal{K}[C]$ is the set of instances of $\mathcal{K}$ in which all terms are subterms of ground terms in either $\mathcal{K}$ or C). Since the size of $\mathcal{K}[C]$ is polynomial in the size of C for a fixed $\mathcal{K}$ and satisfiability of sets of ground Horn clauses can be checked in linear time [9], it follows that for local theories, validity of ground Horn clauses can be checked in polynomial time. Givan and McAllester proved that every problem which is decidable in PTIME can be encoded as an entailment problem of ground clauses w.r.t. a local theory [12]. The property above can be easily generalized to the notion of locality of a set of (Horn) clauses:

Definition 8 *A* local theory *is a set of Horn clauses $\mathcal{K}$ such that, for any set G of ground Horn clauses, $\mathcal{K} \cup G \models \perp$ if and only if already $\mathcal{K}[G] \cup G \models \perp$, where $\mathcal{K}[G]$ is the set of instances of $\mathcal{K}$ in which all terms are subterms of ground terms in either $\mathcal{K}$ or G.*

The same considerations as above can be used to show that in any local theory satisfiability of sets of ground Horn clauses can be checked in polynomial time. In [10], Ganzinger established a link between proof theoretic and semantic concepts for polynomial time decidability of uniform word problems which had already been studied in algebra [18, 7]. In the course of this work he introduced and studied, besides locality, also the less restrictive notion of *stable locality* for equational Horn theories.

Definition 9 *A set $\mathcal{K}$ of Horn clauses is* stably local *if for every set G of ground clauses, if $\mathcal{K} \cup G \models \perp$ then G can be refuted using the set $\mathcal{K}^{[G]}$ of all instances of $\mathcal{K}$ obtained by instantiating the variables with (ground) subterms of G, i.e. if*

$$\mathcal{K} \cup G \models \perp \text{ if and only if } \mathcal{K}^{[G]} \cup G \models \perp .$$

The more general notion of Ψ-stably local theory (in which the instances to be considered are described by a closure operation Ψ) is introduced in [13]. Let $\mathcal{K}$ be a set of clauses. Let $\Psi_{\mathcal{K}}$ be a function associating with any set T of ground terms a set $\Psi_{\mathcal{K}}(T)$ of ground terms such that

(i) all ground subterms in $\mathcal{K}$ and T are in $\Psi_{\mathcal{K}}(T)$;

(ii) for all sets of ground terms T, T' if $T \subseteq T'$ then $\Psi_{\mathcal{K}}(T) \subseteq \Psi_{\mathcal{K}}(T')$;

(iii) for all sets of ground terms T, $\Psi_{\mathcal{K}}(\Psi_{\mathcal{K}}(T)) \subseteq \Psi_{\mathcal{K}}(T)$;

(iv) Ψ is compatible with any map h between constants, i.e. for any map $h : C \to C$, $\Psi_{\mathcal{K}}(\overline{h}(T)) = \overline{h}(\Psi_{\mathcal{K}}(T))$, where $\overline{h}$ is the unique extension of h to terms.

Let $\mathcal{K}^{[\Psi_{\mathcal{K}}(G)]}$ be the set of instances of $\mathcal{K}$ where the variables are instantiated with terms in $\Psi_{\mathcal{K}}(\mathsf{st}(\mathcal{K}, G))$ (set denoted in what follows by $\Psi_{\mathcal{K}}(G)$), where $\mathsf{st}(\mathcal{K}, G)$ is the set of all ground terms occurring in $\mathcal{K}$ or G. We say that $\mathcal{K}$ is Ψ-stably local if it satisfies:

(SLoc^{Ψ}) for every finite set G of ground clauses, $\mathcal{K} \cup G \models \perp$ iff $\mathcal{K}^{[\Psi_{\mathcal{K}}(G)]} \cup G$ has no partial model in which all terms in $\Psi_{\mathcal{K}}(G)$ are defined.

In the particular case when $\Psi_{\mathcal{K}}(G) = \mathsf{st}(\mathcal{K}, G)$ we obtain again the notion of *stable locality* (the corresponding condition is denoted SLoc).

Theorem 4 (Complexity) *If a set $\mathcal{K}$ of Horn clauses satisfies* (SLoc^{Ψ}) *then satisfiability of any set G of Horn clauses w.r.t. $\mathcal{K}$ is decidable in polynomial time in the size of $\Psi_{\mathcal{K}}(G)$.*

Proof: This follows from the fact that $\mathcal{K}^{[\Psi_{\mathcal{K}}(G)]} \cup G$ is a set of ground Horn clauses of size polynomial in the size of $\Psi_{\mathcal{K}}(G)$, and satisfiability of sets of ground Horn clauses (in a relational encoding, taking into account only suitable instances of the congruence axioms – which are also Horn and not more than $|\Psi_{\mathcal{K}}(G)|^2$) can be checked in linear time ([9], see also [10]). □

4.2.1 Recognizing Stably Local Theories

Locality can be recognized by proving embeddability of partial into total models [20, 25, 13]. Theories satisfying (SLoc^{Ψ}) can be recognized by showing that Evans partial models of $\mathcal{T}_1$ embed into total models.

Theorem 5 *Let $\mathcal{K}$ be a set of clauses. Assume $\Psi_{\mathcal{K}}$ satisfies conditions (i)–(iv) above, and that every Evans partial model of $\mathcal{K}$ with the property that the set of defined terms is closed under $\Psi_{\mathcal{K}}$ weakly embeds into a total model of $\mathcal{K}$. Then $\mathcal{K}$ satisfies SLoc^{Ψ}.*

Proof: Let G be a set of ground clauses. We show that, under the given assumptions, if $\mathcal{K} \cup G \models \perp$ then $\mathcal{K}^{[\Psi_{\mathcal{K}}(G)]} \cup G$ has no partial algebra model in which all (ground) terms in $\Psi_{\mathcal{K}}(G)$ are defined. Assume that $\mathcal{K}^{[\Psi_{\mathcal{K}}(G)]} \cup G$ has a partial Evans model P in which all (ground) terms occurring in $\Psi_{\mathcal{K}}(G)$ are defined. We construct a partial model A of $\mathcal{K} \cup G$ as follows. Let $A = \{t_P \mid t \in \Psi_{\mathcal{K}}(G)\}$. As we want A to be a model of $\mathcal{K} \cup G$ in Evans' sense, we need to make sure that if f is an n-ary function and $t^1_P, \dots, t^n_P \in A$ and $(f(t^1, \dots, t^n))_P$ is defined and equal to, say, $t_P \in A$, then $f_A(t^1_P, \dots, t^n_P)$ is defined in A and equal to t_P. Thus, we impose that $f_A(t^1_P, \dots, t^n_P)$ is defined and yields t_P as a result iff $t_P = f(t^1, \dots, t^n)_P \in A$. We show that the set of defined terms in $\mathcal{A}$ is closed under $\Psi_{\mathcal{K}}$. Note first that, by definition of A, for any ground term t, t_A is defined if and only if there exists $t' \in \Psi_{\mathcal{K}}(G)$ such that $t_A = t'_A$. Thus,

$$\mathsf{Def}(A) = \{t \mid t \text{ ground term }, t_A \text{ defined}\} = \overline{h}(\Psi_{\mathcal{K}}(G)),$$

where $\overline{h}$ is the unique homomorphism which extends the map h with $h(c) = c_P$ for every constant c occurring in $\Psi_{\mathcal{K}}(G)$. Then:

$$\Psi_{\mathcal{K}}(\mathsf{Def}(A)) = \Psi_{\mathcal{K}}(\overline{h}(\Psi_{\mathcal{K}}(G))) = \overline{h}(\Psi_{\mathcal{K}}(\Psi_{\mathcal{K}}(G))) \subseteq \overline{h}(\Psi_{\mathcal{K}}(G)) = \mathsf{Def}(A).$$

By condition (i), all ground literals occurring in G are defined in P and (by construction) also in A. Therefore, A satisfies a ground literal L which occurs in G iff P satisfies L. Hence, A satisfies all clauses in G.

It remains to show that A satisfies $\mathcal{K}$. Let $D \in \mathcal{K}$, and $\beta : X \to A$. For every $x \in X$ there exists at least one $t \in \Psi_{\mathcal{K}}(G)$ with $\beta(x) = t_P$. Thus, there exists at least one substitution $\sigma : X \to \Psi_{\mathcal{K}}(G)$ such that $h(\sigma(t)) = \beta(t)$ for all terms t, where h is the canonical projection which associates with every term t its interpretation t_P in P. Then $\sigma(D)$ is an instance of D in $\mathcal{K}^{[\Psi_{\mathcal{K}}(G)]}$. We know that P is a model of $K^{[\Psi_{\mathcal{K}}(G)]}$, hence $(P, h) \models \sigma(D)$. Therefore $(A, \beta) \models D$.

Thus, A satisfies $\mathcal{K} \cup G$. Therefore, A weakly embeds into a total model B of $\mathcal{K}$. It is easy to see that B satisfies the same ground literals as A, so B satisfies all clauses in G. Thus, B is a model of $\mathcal{K} \cup G$, so $\mathcal{K} \cup G \not\models \perp$. □

4.3 Local Theory Extensions

We will also consider extensions of theories, in which the signature is extended by new *function symbols* (i.e. we assume that the set of predicate symbols remains unchanged in the extension).

Let $\mathcal{T}_0$ be an arbitrary theory with signature $\Pi_0 = (S_0, \Sigma_0, \mathsf{Pred})$, where S_0 is a set of sorts, Σ_0 a set of function symbols, and Pred a set of predicate symbols.

We consider extensions $\mathcal{T}_1$ of $\mathcal{T}_0$ with signature $\Pi = (S, \Sigma, \mathsf{Pred})$, where the set of sorts is $S = S_0 \cup S_1$ and the set of function symbols is $\Sigma = \Sigma_0 \cup \Sigma_1$ (i.e. the signature is extended by new sorts and function symbols). We assume that $\mathcal{T}_1$ is obtained from $\mathcal{T}_0$ by adding a set $\mathcal{K}$ of (universally quantified) clauses in the signature Π. Thus, $\mathsf{Mod}(\mathcal{T}_1)$ consists of all Π-structures which are models of $\mathcal{K}$ and whose reduct to Π_0 is a model of $\mathcal{T}_0$.

Terminology. Everywhere in what follows, when referring to *weak (resp. Evans) partial models* of $\mathcal{T}_0 \cup \mathcal{K}$, $\mathcal{T}_0 \cup \mathcal{K}[G]$ resp. $\mathcal{T}_0 \cup \mathcal{K}^{[G]}$ we mean weak (resp. Evans) partial models of *whose reduct to* Π_0 *is a total model of* $\mathcal{T}_0$.

4.3.1 Locality and Stable Locality of an Extension

Let $\mathcal{K}$ be a set of clauses in the signature Π. In what follows, when we refer to sets G of ground clauses we assume that they are in the signature $\Pi^c = (S, \Sigma \cup \Sigma_c, \mathsf{Pred})$, where Σ_c is a set of new constants.

If T is a set of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$-terms, where Σ_c is a set of (new) constants, we denote by $\mathcal{K}_T$ the set of all instances of $\mathcal{K}$ in which all terms starting with a Σ_1-function symbol are ground terms in T. We denote by $\mathcal{K}^T$ the set of all instances of $\mathcal{K}$ in which all variables occurring below a Σ_1-function symbol are instantiated with ground terms in the set $T_{\Sigma_0}(T)$ of Σ_0-terms generated by T.

If G is a set of ground clauses and $T = \mathsf{st}(\mathcal{K}, G)$ is the set of ground subterms occurring in either $\mathcal{K}$ or G then we write $\mathcal{K}[G] := \mathcal{K}_T$, and $\mathcal{K}^{[G]} := \mathcal{K}^T$.

We will focus on the following type of locality of a theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$, where $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$ with $\mathcal{K}$ a set of (universally quantified) clauses:

(Loc) For every finite set G of ground clauses $\mathcal{T}_1 \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ has no weak partial model with all terms in $\mathsf{st}(\mathcal{K}, G)$ defined.

(SLoc) For every set G of ground clauses $\mathcal{T}_1 \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}^{[G]} \cup G$ has no (Evans) partial model with all terms in $\mathsf{st}(\mathcal{K}, G)$ defined.

We say that an extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is *local* if it satisfies condition (Loc). (Note that a local equational theory [10] is a local extension of the pure theory of equality (with no function symbols).) The extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ is *stably local* if it satisfies condition (SLoc).

Notions of Ψ-locality can be defined as in the case of local theories [13]. In Ψ-local theories and theory extensions hierarchical reasoning is possible. We present the ideas for the case of local and stably local theory extensions.

4.3.2 Hierarchical Reasoning

Consider a local theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$. The locality conditions defined above require that, for every set G of ground clauses, $\mathcal{T}_0 \cup \mathcal{K} \cup G$ is satisfiable if and only if $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ has a weak partial model whose Π_0-reduct is a total model of $\mathcal{T}_0$ and in which all the terms in $\mathsf{st}(\mathcal{K}, G)$ are defined.

All clauses in $\mathcal{K}[G] \cup G$ have the property that the function symbols in Σ_1 have as arguments only ground terms. Therefore, $\mathcal{K}[G] \cup G$ can be flattened and purified (i.e. the function symbols in Σ_1 are separated from the other symbols) by introducing, in a bottom-up manner, new constants c_t for subterms $t = f(g_1, \dots, g_n)$ with $f \in \Sigma_1$, g_i ground $\Sigma_0 \cup \Sigma_c$-terms (where Σ_c is a set of constants which contains the constants introduced by flattening, resp. purification), together with corresponding definitions $c_t = t$.

The set of clauses thus obtained has the form $\mathcal{K}_0 \cup G_0 \cup D$, where D is a set of ground unit clauses of the form $f(g_1, \dots, g_n) = c$, where $f \in \Sigma_1$, c is a constant, $g_1, \dots, g_n$ are ground terms without function symbols in Σ_1, and $\mathcal{K}_0$ and G_0 are clauses without function symbols in Σ_1. Flattening and purification preserve both satisfiability and unsatisfiability w.r.t. total algebras, and also w.r.t. partial algebras in which all ground subterms which are flattened are defined [20].

For the sake of simplicity in what follows we will always flatten and then purify $\mathcal{K}[G] \cup G$. Thus we ensure that D consists of ground unit clauses of the form $f(c_1, \dots, c_n) = c$, where $f \in \Sigma_1$, and $c_1, \dots, c_n, c$ are constants. A similar transformation can be performed starting with $\mathcal{K}^{[G]} \cup G$.

Lemma 2 ([20]) *Let $\mathcal{K}$ be a set of clauses. Assume that $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a local (or stably local) theory extension. For any set G of ground clauses, let $\mathcal{K}_0 \cup G_0 \cup D$ be obtained from $\mathcal{K}[G] \cup G$ by flattening and purification, as explained above. Then the following are equivalent:*

(1) $\mathcal{T}_0 \cup \mathcal{K} * [G] \cup G$ has a partial model with all terms in $\mathsf{st}(\mathcal{K}, G)$ defined.

(2) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup D$ has a partial model with all terms in $\mathsf{st}(\mathcal{K}, G)$ defined.

(3) $\mathcal{T}_0 \cup \mathcal{K}_0 \cup G_0 \cup N_0$ has a (total) model, where

$$N_0 = \{\bigwedge_{i=1}^{n} c_i = d_i \rightarrow c = d \mid f(c_1, \dots, c_n) = c, f(d_1, \dots, d_n) = d \in D\}.$$

Here, $\mathcal{K} * [G]$ is $\mathcal{K}[G]$ if $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a local theory extension and $\mathcal{K} * [G]$ is $\mathcal{K}^{[G]}$ if $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a stably local theory extension.

A similar hierarchical reduction can be performed also for Ψ-local (or stably local) theory extensions.

Theorem 6 ([20]) *Assume that the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies either (1) condition* (Loc) *or (2)* (SLoc) *and $\mathcal{T}_0$ is locally finite.*

If all variables in the clauses in $\mathcal{K}$ occur below some function symbol from Σ_1 and if testing satisfiability of ground clauses in $\mathcal{T}_0$ is decidable, then testing satisfiability of ground clauses in $\mathcal{T}_1$ is decidable.

4.3.3 Recognizing Local Theory Extensions

The locality or stable locality of an extension can be recognized by proving embeddability of partial into total models [20, 25, 13]. We will use the following notation:

$\mathsf{PMod}_{\mathsf{w}}^{\mathsf{fd}}(\Sigma_1, \mathcal{T}_1)$	is the class of all weak partial models of $\mathcal{T}_1$ in which the Σ_1-functions are partial and have a finite domain of definition and all function symbols in Σ_0 are total.
$\mathsf{PMod}^{\mathsf{fd}}(\Sigma_1, \mathcal{T}_1)$	is the class of all (Evans) partial models of $\mathcal{T}_1$ in which the Σ_1-functions are partial and have a finite domain of definition and all function symbols in Σ_0 are total.

For theory extensions $\mathcal{T}_0 \subseteq \mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$, where $\mathcal{K}$ is a set of clauses, we consider the following conditions:

($\mathsf{Emb}_{\mathsf{w}}^{\mathsf{fd}}$) Every $A \in \mathsf{PMod}_{\mathsf{w}}^{\mathsf{fd}}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.
($\mathsf{Emb}^{\mathsf{fd}}$) Every $A \in \mathsf{PMod}^{\mathsf{fd}}(\Sigma_1, \mathcal{T}_1)$ weakly embeds into a total model of $\mathcal{T}_1$.

In what follows we say that a non-ground clause is Σ_1-*flat* if function symbols (including constants) do not occur as arguments of function symbols in Σ_1. A Σ_1-flat non-ground clause is called Σ_1-*linear* if whenever a variable occurs in two terms in the clause which start with function symbols in Σ_1, the two terms are identical, and if no term which starts with a function symbol in Σ_1 contains two occurrences of the same variable.

Theorem 7 ([20, 25]) *Let $\mathcal{K}$ be a set of Σ-flat and Σ-linear clauses. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies ($\mathsf{Emb}_{\mathsf{w}}^{\mathsf{fd}}$) then it satisfies ($\mathsf{Loc}$).*

A similar criterion can be given for stably-local theory extensions by combining ideas in the proofs of similar results in [20, 25].

Theorem 8 *Let $\mathcal{T}_0$ be a universal theory and $\mathcal{K}$ be a set of clauses. If the extension $\mathcal{T}_0 \subseteq \mathcal{T}_1$ satisfies ($\mathsf{Emb}^{\mathsf{fd}}$) then it satisfies ($\mathsf{SLoc}$).*

Similar results hold also for Ψ-locality of an extension (cf. e.g. [13, 14]).

5 Locality and Complexity of $\mathcal{EL}^+$ and $\mathcal{EL}$

We now show that the classes of algebraic models of $\mathcal{EL}^+$ and of $\mathcal{EL}$ have presentations which satisfy certain locality properties. This gives an alternative, algebraic explanation of the fact that CBox subsumption in these logics is decidable in PTIME and makes generalizations possible.

5.1 Locality and $\mathcal{EL}^+$

In this section we prove that the class $\mathsf{SLO}_\Sigma(RI)$ of semilattices with monotone operators in a set Σ satisfying a family RI_a of axioms of the form

$$\forall x\, f_1(\dots(f_n(x))\dots) \leq f(x)$$

has a local presentation, and therefore the uniform word problem w.r.t. this class can be decided in polynomial time. For the sake of simplicity we restrict, w.l.o.g., to axioms as above with $n \in \{1, 2\}$.

It is known that the theories of lattices and semilattices allow local Horn axiomatizations (cf. e.g. [18, 7]). Let SL be such an axiomatization for the theory of semilattices. We denote by $\mathsf{Mon}(\Sigma)$ the set $\{\mathsf{Mon}(f) \mid f \in \Sigma\}$, where

$$\mathsf{Mon}(f) \quad \forall x, y(x \leq y \to f(x) \leq f(y)).$$

Theorem 9 *The set of Horn clauses $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$ has the property that every Evans partial model A with the properties:*

(i) for every $f \in \Sigma$, f_A is a partial function with finite definiton domain;

(ii) for each axiom in RI_a of the form $f_1(f_2(x)) \leq f(x)$, and every $a \in A$, if $f(a)$ is defined then $f_2(a)$ is defined[5] in A;

(iii) $A \models SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$;

weakly embeds into a total model of $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$.

Proof: Let A be an Evans partial model of $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$ with properties (i)–(iii). In particular, A is a poset, hence it embeds into a complete (semi)lattice S such that the meets that exist in A are preserved. (We will think of A as a subset of S.) For every $f \in \Sigma$ we define $\overline{f} : S \to S$ by

$$\overline{f}(a) = \bigwedge\{f(c) \mid a \leq c, c \in A, f_A(c) \text{ is defined}\}.$$

For every $f \in \Sigma$, $\overline{f}$ is monotone (see e.g. also [25]). We show that the axioms in RI_a are satisfied by these extensions.

- Let $f_1(x) \leq f_2(x) \in RI_a$ and $a \in S$. Then: $\overline{f}_i(a) = \bigwedge\{f_i(c) \mid a \leq c, c \in A, f_i(c) \text{ is defined}\}$. Let $d \in A$ with $a \leq d$ and $f_2(d)$ defined. Then $f_1(d)$ is also defined and $f_1(d) \leq f_2(d)$. Thus, $\overline{f}_1(a) \leq f_2(d)$ for all $d \in A$ with $a \leq d$ and $f_2(d)$ defined, so $\overline{f}_1(a) \leq \overline{f}_2(a)$.

[5] From Definitions 4 and 5 it follows that then also $f_1(f_2(a))$ is defined.

- Let now $f_1(f_2(x)) \leq f(x) \in RI_a$ and $a \in S$. Then $\overline{f}_2(a) = \bigwedge\{f_2(c) \mid a \leq c, c \in A, f_2(c)$ is defined$\}$. Then for every $a \leq c$, if $f_2(c)$ is defined then $\overline{f}_2(a) \leq f_2(c)$. We prove that $\overline{f}_1(\overline{f}_2(a)) \leq \overline{f}(a)$.

  Note first that if $a \leq c$ and $f_1(f_2(c))$ is defined then $\overline{f}_2(a) \leq f_2(c)$. Therefore, $f_1(f_2(c)) \in \{f_1(c_1) \mid \overline{f}_2(a) \leq c_1,$ and $f_1(c_1)$ defined$\}$. Hence, $\{f_1(f_2(c)) \mid a \leq c, f_1(f_2(c))$ defined$\} \subseteq \{f_1(c_1) \mid \overline{f}_2(a) \leq c_1, f_1(c_1)$ defined$\}$. Therefore, the infimum of the first set is larger than the infimum of the second set. Hence:

$$\begin{array}{rcl}\overline{f}_1(\overline{f}_2(a)) & = & \bigwedge\{f_1(c_1) \mid \overline{f}_2(a) \leq c_1, f_1(c_1) \text{ is defined}\} \\ & \leq & \bigwedge\{f_1(f_2(c)) \mid a \leq c \text{ and } f_1(f_2(c)) \text{ defined}\} \\ & \leq & \bigwedge\{f(c) \mid a \leq c \text{ and } f(c) \text{ defined}\} = \overline{f}(a).\end{array}$$

The last inequality is a consequence of the fact that if $f(d)$ is defined in A then $f_2(d)$ is defined in A, and since $A \models RI_a$, $f_1(f_2(d))$ is defined in A and $f_1(f_2(d)) \leq f(d)$. Hence, $\bigwedge\{f_1(f_2(c)) \mid a \leq c$ and $f_1(f_2(c))$ defined$\} \leq f_1(f_2(d)) \leq f(d)$. □

Corollary 1 *The following are equivalent:*

(1) $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a \models \forall \overline{x} \bigwedge_{i=1}^{n} s_i(\overline{x}) \leq s'_i(\overline{x}) \rightarrow s(\overline{x}) \leq s'(\overline{x})$*;*

(2) $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a \cup G \models \perp$*, where* $G = \bigwedge_{i=1}^{n} s_i(\overline{c}) \leq s'_i(\overline{c}) \wedge s(\overline{c}) \not\leq s'(\overline{c})$*;*

(3) $(SL \cup \mathsf{Mon}(\Sigma) \cup RI_a)^{[\Psi_{RI}(G)]} \cup G \models \perp$ *where* $\Psi_{RI}(G) = \bigcup_{i \geq 0} \Psi^i_{RI}$*, with* $\Psi^0_{RI} = \mathsf{st}(G)$*, and* $\Psi^{i+1}_{RI} = \{f_2(d) \mid f(d) \in \Psi^i_{RI}, (f_1 \circ f_2)(x) \leq f(x) \in RI_a\}$*.*

Here $\mathsf{st}(G)$ is the set of all (ground) subterms occurring in G. Note that $\Psi_{RI}(G)$ can have at most $|\mathsf{st}(G)| \cdot |N_R|$ elements. Thus, its size is polynomial in the size of G. On the other hand, the number of clauses in $(SL \cup \mathsf{Mon}(\Sigma) \cup RI_a)^{[\Psi_{RI}(G)]}$ is polynomial in $|\Psi_{RI}(G)|$, and satisfiability of any set of ground clauses can be tested in polynomial time. This shows that the uniform word problem for the class $\mathsf{SLO}_\Sigma(RI)$ (and thus also for $\mathsf{SLO}^{\exists}_{NR}(RI)$) is decidable in polynomial time.

Example 1 *We illustrate the ideas on an example presented in [4] (here slightly simplified). Consider the CBox C consisting of the following GCI:*

$$\begin{aligned}
\mathsf{Endocard} &\sqsubseteq \mathsf{Tissue} \sqcap \exists \mathsf{cont\text{-}in}.\mathsf{HeartWall} \sqcap \exists \mathsf{cont\text{-}in}.\mathsf{HeartValve}\\
\mathsf{HeartWall} &\sqsubseteq \exists \mathsf{part\text{-}of}.\mathsf{Heart}\\
\mathsf{HeartValve} &\sqsubseteq \exists \mathsf{part\text{-}of}.\mathsf{Heart}\\
\mathsf{Endocarditis} &\sqsubseteq \mathsf{Inflammation} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{Endocard}\\
\mathsf{Inflammation} &\sqsubseteq \mathsf{Disease}\\
\mathsf{Heartdisease} &= \mathsf{Disease} \sqcap \exists \mathsf{has\text{-}loc}.\mathsf{Heart}
\end{aligned}$$

and the following role inclusions RI:

$$\begin{aligned}
\mathsf{cont\text{-}in} \circ \mathsf{cont\text{-}in} &\sqsubseteq \mathsf{cont\text{-}in}\\
\mathsf{part\text{-}of} &\sqsubseteq \mathsf{cont\text{-}in}\\
\mathsf{has\text{-}loc} \circ \mathsf{cont\text{-}in} &\sqsubseteq \mathsf{has\text{-}loc}
\end{aligned}$$

We want to check whether Endocarditis $\sqsubseteq_{\mathcal{C}}$ Heartdisease*. This is the case iff (with some abbreviations – e.g.* f_{ci} *stands for* $f_{\exists \mathsf{cont\text{-}in}}$ *and* f_{po} *for* $f_{\exists \mathsf{part\text{-}of}}$*,* h_w *and* h_v *for* HeartWall *resp.* HeartValve*,* e *for* Endocard*,* h *for* Heart*, etc.):*

$$\begin{aligned}
SL \;\cup\; & \mathsf{Mon}(f_{\mathsf{ci}}, f_{\mathsf{hl}}, f_{\mathsf{po}}) \;\cup\; \{\forall x\ f_{\mathsf{ci}}(f_{\mathsf{ci}}(x)) \leq f_{\mathsf{ci}}(x),\\
& \qquad \forall x\ f_{\mathsf{po}}(x) \leq f_{\mathsf{ci}}(x),\\
& \qquad \forall x\ f_{\mathsf{hl}}(f_{\mathsf{ci}}(x)) \leq f_{\mathsf{hl}}(x)\}\\
\cup\; & \{e \leq t \wedge f_{\mathsf{ci}}(h_w) \wedge f_{\mathsf{ci}}(h_v), h_w \leq f_{\mathsf{po}}(h), \quad h_v \leq f_{\mathsf{po}}(h),\\
& \mathsf{Endocarditis} \leq i \wedge f_{\mathsf{hl}}(e), \quad i \leq d, \quad \mathsf{Heartdisease} = d \wedge f_{\mathsf{hl}}(h),\\
& \mathsf{Endocarditis} \not\leq \mathsf{Heartdisease}\} \quad \models \quad \perp .
\end{aligned}$$

Then $\mathsf{st}(\mathcal{K}, G) = \{f_{\mathsf{ci}}(h_w), f_{\mathsf{ci}}(h_v), f_{\mathsf{po}}(h), f_{\mathsf{hl}}(e), f_{\mathsf{hl}}(h)\}$*. To compute* $\Psi_{\mathcal{K}}(G)$*, note that* $\Psi^0_{RI} = \mathsf{st}(\mathcal{K}, G)$*,* $\Psi^1_{RI} = \{f_{\mathsf{ci}}(e), f_{\mathsf{ci}}(h)\}$*, and* $\Psi^2_{RI} = \Psi^1_{RI}$*.*
Thus, $\Psi_{\mathcal{K}}(G) = \{f_{\mathsf{ci}}(h_w), f_{\mathsf{ci}}(h_v), f_{\mathsf{ci}}(e), f_{\mathsf{ci}}(h), f_{\mathsf{po}}(h), f_{\mathsf{hl}}(e), f_{\mathsf{hl}}(h)\}$*.*
After computing $(RI_a \cup \mathsf{Mon}(f_{\mathsf{ci}}, f_{\mathsf{hl}}, f_{\mathsf{po}}) \cup \mathsf{Con})^{[\Psi(G)]}$ *and* $SL^{[\Psi(G)]}$ *we obtain the following conjunction of (Horn) ground clauses:*

G	$(RI_a \wedge \mathsf{Mon} \wedge \mathsf{Con})^{[\Psi(G)]} \wedge SL^{[\Psi(G)]}$
$e \leq t \wedge f_{\mathsf{ci}}(h_w) \wedge f_{\mathsf{ci}}(h_v)$	$f_{\mathsf{ci}}(f_{\mathsf{ci}}(x)) \leq f_{\mathsf{ci}}(x)$ *for* $x \in \Psi_{\mathcal{K}}(G)$
$h_w \leq f_{\mathsf{po}}(h)$	$f_{\mathsf{po}}(x) \leq f_{\mathsf{ci}}(x)$ *for* $x \in \Psi_{\mathcal{K}}(G)$
$h_v \leq f_{\mathsf{po}}(h)$	$f_{\mathsf{hl}}(f_{\mathsf{ci}}(x)) \leq f_{\mathsf{hl}}(x)$ *for* $x \in \Psi_{\mathcal{K}}(G)$
$\mathsf{Endocarditis} \leq i \wedge f_{\mathsf{hl}}(e)$	
$i \leq d$	$xRy \rightarrow f_{\mathsf{ci}}(x) R f_{\mathsf{ci}}(y)$ *for* $x, y \in \Psi_{\mathcal{K}}(G)$
$\mathsf{Heartdisease} = d \wedge f_{\mathsf{hl}}(h)$	$xRy \rightarrow f_{\mathsf{po}}(x) R f_{\mathsf{po}}(y)$ *for* $x, y \in \Psi_{\mathcal{K}}(G)$
$\mathsf{Endocarditis} \not\leq \mathsf{Heartdisease}$	$xRy \rightarrow f_{\mathsf{hl}}(x) R f_{\mathsf{hl}}(y)$ *for* $x, y \in \Psi_{\mathcal{K}}(G)$
	$R \in \{\leq, \geq, =\}$
	$SL^{[\Psi(G)]}$

By Corollary 1, $\mathsf{Endocarditis} \sqsubseteq_{\mathcal{C}} \mathsf{Heartdisease}$ *iff* $\phi = G \wedge (RI_a \wedge \mathsf{Mon} \wedge \mathsf{Con})^{[\Psi(G)]} \wedge SL^{[\Psi(G)]}$ *is unsatisfiable. Note that* ϕ *is a set of ground clauses in first-order logic with equality, containing all instances of the congruence axioms corresponding to the (ground) terms which occur in* ϕ.

A translation to Datalog can easily be obtained by replacing the function symbols with binary predicate symbols. Alternatively, we can process the instances in ϕ *by replacing, in a bottom-up fashion, all the terms starting with function symbols (which are all ground) with new constants (and adding, separately, the corresponding definitions) (cf. e.g. the remarks in [10, 7]).*

The satisfiability of ϕ *can therefore be checked automatically in polynomial time in the size of* ϕ *which in its turn is polynomial in the size of* $\Psi_{\mathcal{K}}(G)$. *Hence, in this case, the size of* ϕ *is polynomial in the size of* G.

Note however that if $\Psi(G)$ *has 7 elements, the size of* $(RI_a \wedge \mathsf{Mon} \wedge \mathsf{Con})^{[\Psi(G)]}$ *will be 168. The number of instances of clauses in* $SL^{[\Psi(G)]}$ *is* $c \cdot 7^3$ *where* c *is a constant. Thus, although the number of instances which we need to consider is polynomial in the size of* G, *it can sometimes be large.*

Unsatisfiability can also be proved directly: G *entails the inequalities:*

(1) $\mathsf{Endocarditis} \leq (d \wedge f_{\mathsf{hl}}(e))$; (2) $e \leq (f_{\mathsf{ci}}(h_w) \wedge f_{ci}(h_v))$;

(3) $(h_w \leq f_{\mathsf{po}}(h))$; (4) $(h_v \leq f_{\mathsf{po}}(h))$.

Hence $G \wedge (RI_a \wedge \mathsf{Mon} \wedge \mathsf{Con})^{[\Psi(G)]} \models e \leq f_{\mathsf{ci}}(f_{\mathsf{po}}(h)) \leq f_{\mathsf{ci}}(f_{\mathsf{ci}}(h)) \leq f_{\mathsf{ci}}(h)$. *Thus,* $G \wedge (RI_a \wedge \mathsf{Mon} \wedge \mathsf{Con})^{[\Psi(G)]} \models f_{\mathsf{hl}}(e) \leq f_{\mathsf{hl}}(f_{\mathsf{ci}}(h)) \leq f_{\mathsf{hl}}(h)$, *so* $G \wedge (RI_a \wedge \mathsf{Mon} \wedge \mathsf{Con})^{[\Psi(G)]} \models \mathsf{Endocarditis} \leq d \wedge f_{\mathsf{hl}}(h)$, *which together with* $d \wedge f_{\mathsf{hl}}(h) = \mathsf{Heartdisease}$ *and* $\mathsf{Endocarditis} \not\leq \mathsf{Heartdisease}$ *leads to a contradiction.*

5.2 Locality and $\mathcal{EL}$

In [19] we proved that the algebraic counterpart of the description logic $\mathcal{EL}$, namely the class of semilattices with monotone operators – axiomatized by $SL \cup \mathsf{Mon}(\Sigma)$ – has an even stronger locality property, namely for every set G of ground clauses

$$SL \cup \mathsf{Mon}(\Sigma) \wedge G \models \perp \quad \text{if and only if} \quad (SL \cup \mathsf{Mon}(\Sigma))[G] \wedge G \models \perp$$

where $\mathcal{K}[G]$ is the set of instances of $\mathcal{K}$ containing only ground terms occurring in G. In fact, we showed that the extension of the theory SL of semilattices with a family of monotone functions is local in the sense defined in [20].

Theorem 10 ([25]) *Let G be a set of ground clauses. The following are equivalent:*

(1) $SL \cup \mathsf{Mon}(\Sigma) \cup G \models \perp$.

(2) $SL \cup \mathsf{Mon}(\Sigma)[G] \cup G$ *has no partial model A such that its $\{\wedge\}$-reduct is a (total) semilattice and the functions in Σ are partially defined, their domain of definition is finite and all terms in G are defined in A.*

Let $\mathsf{Mon}(\Sigma)[G]_0 \cup G_0 \cup \mathsf{Def}$ *be obtained from* $\mathsf{Mon}(\Sigma)[G] \cup G$ *by purification, i.e. by replacing, in a bottom-up manner, all subterms $f(g)$ with $f \in \Sigma$, with newly introduced constants $c_{f(g)}$ and adding the definitions $f(g) = c_t$ to the set* Def. *The following are equivalent (and equivalent to (1) and (2)):*

(3) $\mathsf{Mon}(\Sigma)[G]_0 \cup G_0 \cup \mathsf{Def}$ *has no partial model $(A, \wedge, \{f_A\}_{f \in \Sigma})$ such that $(A, \wedge)$ is a semilattice and for all $f \in \Sigma$, f_A is partially defined, its domain of definition is finite and all terms in* Def *are defined in A;*

(4) $\mathsf{Mon}(\Sigma)[G]_0 \cup G_0$ *is unsatisfiable in SL.*

(Note that in the presence of $\mathsf{Mon}(\Sigma)$ *the instances* $\mathsf{Con}[G]_0$ *of the congruence axioms for the functions in Σ are not necessary.)*

$$\mathsf{Con}[G]_0 = \{g{=}g' \rightarrow c_{f(g)}{=}c_{f(g')} \mid f(g){=}c_{f(g)}, f(g'){=}c_{f(g')} \in \mathsf{Def}\}.$$

This equivalence allows us to hierarchically reduce, in polynomial time, proof tasks in $SL \cup \mathsf{Mon}(\Sigma)$ to proof tasks in SL (cf. e.g. [25]) which can then be solved in polynomial time. [6]

Example 2 *We illustrate the method on an example first considered in [2]. Consider the $\mathcal{EL}$ TBox $\mathcal{T}$ consisting of the following definitions:*

$$\begin{array}{rcl} A_1 & = & P_1 \sqcap A_2 \sqcap \exists r_1.\exists r_2.A_3 \\ A_2 & = & P_2 \sqcap A_3 \sqcap \exists r_2.\exists r_1.A_1 \\ A_3 & = & P_3 \sqcap A_2 \sqcap \exists r_1.(P_1 \sqcap P_2) \end{array}$$

[6] We could prove a similar theorem in the presence of role inclusion axioms for certain types of role inclusions. An extension to general role inclusions – which would provide more efficient instantiations, and therefore more efficient algorithms than those provided by Corollary 1 – is subject of work in progress.

We want to prove that $P_3 \sqcap A_2 \sqcap \exists r_1.(A_1 \sqcap A_2) \sqsubseteq_{\mathcal{T}} A_3$. *We translate this subsumption problem to the following satisfiability problem:*

$$\begin{aligned} \mathsf{SL} \cup \mathsf{Mon}(f_1, f_2) \quad \cup \quad \{ & a_1 = (p_1 \wedge a_2 \wedge f_1(f_2(a_3))), \\ & a_2 = (p_2 \wedge a_3 \wedge f_2(f_1(a_1))), \\ & a_3 = (p_3 \wedge a_2 \wedge f_1(p_1 \wedge p_2)), \\ & \neg(p_3 \wedge a_2 \wedge f_1(a_1 \wedge a_2) \leq a_3)\} \models \perp . \end{aligned}$$

We proceed as follows: We flatten and purify the set G *of ground clauses by introducing new names for the terms starting with the function symbols* f_1 *or* f_2. *Let* Def *be the corresponding set of definitions. We then take into account only those instances of the monotonicity and congruence axioms for* f_1 *and* f_2 *which correspond to the instances in* Def, *and purify them as well, by replacing the terms themselves with the constants which denote them. We obtain the following separated set of formulae:*

Def	$G_0 \wedge$	$(\mathsf{Mon}(f_1, f_2)[G])_0$
$f_2(a_3) = c_1$	$(a_1 = p_1 \wedge a_2 \wedge c_2)$	$a_1 R c_1 \rightarrow c_3 R c_2, \quad R \in \{\leq, \geq\}$
$f_1(c_1) = c_2$	$(a_2 = p_2 \wedge a_3 \wedge c_4)$	$a_3 R c_3 \rightarrow c_1 R c_4, \quad R \in \{\leq, \geq\}$
$f_1(a_1) = c_3$	$(a_3 = p_3 \wedge a_2 \wedge d_1)$	$a_1 R e_1 \rightarrow c_3 R d_1, \quad R \in \{\leq, \geq\}$
$f_2(c_3) = c_4$	$(p_3 \wedge a_2 \wedge d_2 \not\leq a_3)$	$a_1 R e_2 \rightarrow c_3 R d_2, \quad R \in \{\leq, \geq\}$
$f_1(e_1) = d_1$	$p_1 \wedge p_2 = e_1$	$c_1 R e_1 \rightarrow c_2 R d_1, \quad R \in \{\leq, \geq\}$
$f_1(e_2) = d_2$	$a_1 \wedge a_2 = e_2$	$c_1 R e_2 \rightarrow c_2 R d_2, \quad R \in \{\leq, \geq\}$
		$e_1 R e_2 \rightarrow d_1 R d_2, \quad R \in \{\leq, \geq\}$

The subsumption is true iff $G_0 \wedge (\mathsf{Mon}(f_1, f_2)[G])_0$ *is unsatisfiable in the theory of semilattices. We can see this as follows: note that* $a_1 \wedge a_2 \leq p_1 \wedge p_2$, *i.e.* $e_2 \leq e_1$. *Then (using an instance of monotonicity)* $d_2 \leq d_1$, *so* $p_3 \wedge a_2 \wedge d_2 \leq p_3 \wedge a_2 \wedge d_1 = a_3$.

This can also be checked automatically in PTIME either by using the fact that there exists a local presentation of SL *or using the fact that* $\mathsf{SL} = ISP(S_2)$ *(i.e. every semilattice is isomorphic with a sublattice of a power of* S_2*), where* S_2 *is the semilattice with two elements, hence* SL *and* S_2 *satisfy the same Horn clauses. Since the theory of semilattices is convex, satisfiability of ground clauses w.r.t.* SL *can be reduced to SAT solving.*

6 Extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$

The results described in Section 5 can easily be generalized to semilattices with n-ary monotone functions satisfying composition axioms. This allows

Table 2: Constructors for $\mathcal{EL}$ with n-ary roles and their semantics

Constructor	Syntax	Semantics
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
existential	$\exists R.(C_1, \dots C_n)$	$\{x \mid \exists y_1, \dots, y_n\, (x, y_1, \dots, y_n) \in R^{\mathcal{I}}$ and $y_i \in C_i^{\mathcal{I}}\}$

us to define natural generalizations of $\mathcal{EL}$ and $\mathcal{EL}^+$. We start by presenting a generalization of $\mathcal{EL}$ in which n-ary roles are allowed. We then sketch possible extensions in which role inclusions are also taken into account.

6.1 Extensions of $\mathcal{EL}$

6.1.1 Extensions with n-ary Roles

We consider extensions of $\mathcal{EL}$ with n-ary roles. The semantics is defined in terms of interpretations $\mathcal{I} = (D^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $D^{\mathcal{I}}$ is a non-empty set, concepts are interpreted as usual, and each n-ary role $R \in N_R$ is interpreted as an n-ary relation $R^{\mathcal{I}} \subseteq (D^{\mathcal{I}})^n$ (cf. Table 2).

The possibility of considering n-ary roles can help in modeling certain notions more precisely than in situations in which we are restricted to binary roles. In medical ontologies, for instance, it is important to be able to express facts of the form *"Disorder D has morphology M at site S"*, modeled using a ternary relation morphology-at-site as morphology-at-site(D, M, S) [28].

6.1.2 Extensions with n-ary Roles and Numerical Domains

A further extension is obtained by allowing for certain concrete sorts – having the same support in all interpretations; or additionally assuming that there exist specific concrete concepts which have a fixed semantics (or additional fixed properties) in all interpretations. The extensions we consider are different from the extensions with concrete domains and those with n-ary quantifiers studied in the description logic literature (cf. e.g. [5, 3]).

Example 3 *Consider a description logic having a usual (*concept*) sort and a 'concrete' sort* num *with fixed domain $\mathbb{N}$. We may be interested in general concrete concepts of sort* num *(interpreted as subsets of $\mathbb{R}$) or in special concepts of sort* num *such as $\uparrow n$, $\downarrow n$, or $[n, m]$ for $m, n \in \mathbb{R}$. For any* *interpretation* $\mathcal{I}$, $\uparrow n^{\mathcal{I}} = \{x \in \mathbb{R} \mid x \geq n\}$, $\downarrow n^{\mathcal{I}} = \{x \in \mathbb{R} \mid x \leq n\}$, *and* $[n,m]^{\mathcal{I}} = \{x \in \mathbb{R} \mid n \leq x \leq m\}$. *We will denote the arities of roles using a many-sorted framework. Let* $(D, \mathbb{R}, \cdot^{\mathcal{I}})$ *be an interpretation with two sorts* concept *and* num. *A role with arity* $(s_1, \dots, s_n)$ *is interpreted as a subset of* $D_{s_1} \times \dots \times D_{s_n}$*, where* $D_{\mathsf{concept}} = D$ *and* $D_{\mathsf{num}} = \mathbb{R}$.

1. *Let* price *be a binary role of arity* (concept, num)*, which associates with every element of sort* concept *its possible prices. The concept*

$$\exists \mathsf{price}.\!\uparrow\! n = \{x \mid \exists k \geq n : \mathsf{price}(x,k)\}$$

represents the class of all individuals with some price *greater than* n.

2. *Let* has-weight-price *be a role of arity* (concept, num, num)*. The concept*

$$\exists\ \mathsf{has\text{-}weight\text{-}price}.(\uparrow y, \downarrow p) = \{x \mid \exists y' {\geq} y, \exists p' {\leq} p \text{ and } \mathsf{has\text{-}weight\text{-}price}(x, y', p')\}$$

denotes the family of individuals for which a weight above y *and a price below* p *exist.*

The example above can be generalized by allowing a set of concrete sorts. We discuss the algebraic semantics of this type of extensions of $\mathcal{EL}$.

Let $\mathsf{SLO}^{\exists}_{N_R,S}$ denote the class of all structures

$$(S, \mathcal{P}(A_1), \dots, \mathcal{P}(A_n), \{f_{\exists r} \mid r \in N_R\}),$$

where S is a semilattice, $A_1, \dots, A_n$ are concrete domains, and $\{f_{\exists r} \mid r \in N_R\}$ are n-ary monotone operators. We may allow constants of concrete sort, interpreted as sets in $\mathcal{P}(A_i)$. The classes $\mathsf{DLO}^{\exists}_{N_R,S}$ and $\mathsf{BAO}^{\exists}_{N_R,S}$ of all distributive lattices resp. Boolean algebras with concrete supports and n-ary join hemimorphisms $\{f_{\exists r} | r \in N_R\}$ are defined similarly.

Theorem 11 *If the only concept constructors are intersection and existential restriction, then for all concept descriptions* D_1, D_2*, and every TBox* $\mathcal{T}$ *consisting of general concept inclusions GCI the following are equivalent:*

(1) $D_1 \sqsubseteq_{\mathcal{T}} D_2$.

(2) $\mathsf{SLO}^{\exists}_{N_R,S} \models \forall C_1, \dots, C_n \left(\left(\bigwedge_{C \sqsubseteq D \in GCI} \overline{C} \leq \overline{D} \right) \rightarrow \overline{D_1} \leq \overline{D_2} \right)$.

Proof: Analogous to the proof of Theorem 3. □

Let SL_S be the class of all structures $\mathcal{A} = (A, \mathcal{P}(A_1), \dots, \mathcal{P}(A_n))$, with signature $\Pi = (S, \{\wedge\} \cup \Sigma, \mathsf{Pred})$ with $S = \{\mathsf{concept}, \mathsf{s_1}, \dots, \mathsf{s_n}\}$, $\mathsf{Pred} = \{\leq\} \cup \{\subseteq_i \mid 1 \leq i \leq n\}$, where $A \in SL$, the support of sort $\mathsf{concept}$ of $\mathcal{A}$ is A, and for all i the support sort s_i of $\mathcal{A}$ is $\mathcal{P}(A_i)$.

Theorem 12 ([25]) *Every structure* $(A, \mathcal{P}(A_1), \dots, \mathcal{P}(A_n), \{f_A\}_{f \in \Sigma})$, *where*

(i) $(A, \mathcal{P}(A_1), \dots, \mathcal{P}(A_n)) \in SL_S$, *and*

(ii) for every $f \in \Sigma$ *of arity* $s_1 \dots s_n \rightarrow s$, f_A *is a partial function from* $\prod_{i=1}^n U_{s_i}$ *to* U_s *with a finite definition domain on which it is monotone ((where* $U_{\mathsf{concept}} = A$ *and* $U_{\mathsf{s}_i} = \mathcal{P}(A_i)$*)),*

weakly embeds into a total model of $SLO_{\Sigma,S}$ *(axiomatized by* $SL_S \cup \mathsf{Mon}(\Sigma)$*).*

Corollary 2 *Let* $G = \bigwedge_{i=1}^n s_i(\overline{c}) \leq s'_i(\overline{c}) \wedge s(\overline{c}) \not\leq s'(\overline{c})$ *be a set of ground unit clauses in the extension* Π^c *of* Π *with new constants* Σ_c*. The following are equivalent:*

(1) $SL_S \cup \mathsf{Mon}(\Sigma) \cup G \models \perp$.

(2) $SL_S \cup \mathsf{Mon}(\Sigma)[G] \cup G$ *has no partial model with a total* $\{\wedge_{SL}\}$*-reduct in which all terms in* G *are defined.*

Let $\bigcup_{i=0}^n \mathsf{Mon}(\Sigma)[G]_i \cup G_i \cup \mathsf{Def}$ *be obtained from* $\mathsf{Mon}(\Sigma)[G] \cup G$ *by purification, i.e. by replacing, in a bottom-up manner, all subterms* $f(g)$ *of sort* s *with* $f \in \Sigma$*, with newly introduced constants* $c_{f(g)}$ *of sort* s *and adding the definitions* $f(g) = c_t$ *to the set* Def*. We thus separate* $\mathsf{Mon}(\Sigma)[G] \cup G$ *into a conjunction of constraints* $\Gamma_i = \mathsf{Mon}(\Sigma)[G]_i \cup G_i$*, where* Γ_0 *is a constraint of sort* $\mathsf{semilattice}$ *and for* $1 \leq i \leq n$*,* Γ_i *is a set of constraints over terms of sort* i *(*i *being the concrete sort with fixed support* $\mathcal{P}(A_i)$*). Then the following are equivalent (and are also equivalent to (1) and (2)):*

(3) $\bigcup_{i=0}^n (\mathsf{Mon}(\Sigma)[G]_i \cup G_i) \cup \mathsf{Def}$ *has no partial model with a total* $\{\wedge_{SL}\}$*-reduct in which all terms in* Def *are defined.*

(4) $\bigcup_{i=0}^n (\mathsf{Mon}(\Sigma)[G]_i \cup G_i)$ *is unsatisfiable in the many-sorted disjoint combination of* SL *and the concrete theories of* $\mathcal{P}(A_i)$*,* $1 \leq i \leq n$*.*

The complexity of the uniform word problem of $SL_S \cup \mathsf{Mon}(\Sigma)$ depends on the complexity of the problem of testing the satisfiability — in the many-sorted disjoint combination of SL with the concrete theories of $\mathcal{P}(A_i)$, $1 \leq i \leq n$ — of sets of clauses $C_{\mathsf{concept}} \cup \bigcup_{i=1}^{n} C_i \cup \mathsf{Mon}$, where C_{concept} and C_i are unit clauses of sort concept resp. s_i, and Mon consists of possibly mixed ground Horn clauses.

Specific extensions of the logic $\mathcal{EL}$ can be obtained by imposing additional restrictions on the interpretation of the "concrete"-type concepts within $\mathcal{P}(A_i)$. For instance, we can require that numerical concepts are always interpreted as intervals, as in Example 3.

A tractable fragment. In what follows we analyze the situation in which the concepts of sort num are described using the Ord-Horn fragment of Allen's interval arithmetic [17] (fragment which has the property that all operations and relations between intervals can be represented by Ord-Horn clauses, i.e. clauses over atoms $x \leq y, x = y$, containing at most one positive literal ($x \leq y$ or $x = y$) and arbitrarily many negative literals (of the form $x \neq y$)).

Theorem 13 *Consider the following extensions of $\mathcal{EL}$ with n-ary roles:*

(1) The one-sorted extension of $\mathcal{EL}$ with n-ary roles.

(2) The extension of $\mathcal{EL}$ with two sorts, concept *and* num*, where the semantics of classical concepts is the usual one, and the concepts of sort* num *are interpreted as elements in the Ord-Horn, convex fragment of Allen's interval algebra [17], where any TBox can contain many-sorted GCI's over concepts, as well as constraints over the numerical data expressible in the Ord-Horn fragment.*

In both cases, TBox subsumption is decidable in PTIME.

Proof: The fact that CBox subsumption is decidable in PTIME in case (1) is an immediate consequence of results in [25].

We prove that this holds also for case (2). The assumption on the semantics of the extension of $\mathcal{EL}$ we made ensures that all algebraic models are two-sorted structures of the form

$$\mathcal{A} = ((A, \wedge), Int(\mathbb{R}, O), \{f_{\mathcal{A}}\}_{f \in \Sigma}),$$

with sorts $\{\mathsf{concept}, \mathsf{num}\}$, such that $(A, \wedge)$ is a semilattice, $Int(\mathbb{R}, O)$ is an interval algebra in the Ord-Horn fragment of Allen's interval arithmetic

[17], and for all $f \in \Sigma$, f_A is a monotone (many-sorted) function. We will denote the class of all these structures by SL_{OrdHorn}. The fact that TBox subsumption is decidable in PTIME now follows from the following facts:

- The theory of semilattices is convex;
- Nebel and Bürckert [17] proved that a finite set of Ord-Horn clauses is satisfiable over the real numbers iff it is satisfiable over posets. As the theory of partial orders is convex, this means that although the theory of reals is not convex w.r.t. $\leq$, we can always assume that the theory of Ord-Horn clauses is convex.

The main result in Corollary 2 can be adapted without problems to show that if $G = \bigwedge_{i=1}^{n} s_i(\overline{c}) \leq s'_i(\overline{c}) \wedge s(\overline{c}) \not\leq s'(\overline{c})$ is a set of ground unit clauses in the extension Π^c of Π with new constants Σ_c, and if $\mathsf{Mon}(\Sigma)[G]_{\mathsf{c}} \cup \mathsf{Mon}(\Sigma)[G]_{\mathsf{num}} \cup G_{\mathsf{c}} \cup G_{\mathsf{num}} \cup \mathsf{Def}$ are obtained from $\mathsf{Mon}(\Sigma)[G] \cup G$ by purification, the following are equivalent:

- $SL_{\mathsf{OrdHorn}} \cup \mathsf{Mon}(\Sigma) \cup G \models \perp$;
- $\mathsf{Mon}(\Sigma)[G]_0 \cup G_0 \cup \mathsf{Con}[\mathsf{Def}]_0$ is unsatisfiable in the combination of SL and the Ord-Horn fragment of Allen's interval arithmetic.

In order to test the unsatisfiability of the latter problem we proceed as follows. We first note that, due to the convexity of the theories involved and to the fact that all constraints in $G_0 \cup \mathsf{Mon}(\Sigma)[G]_0 \cup \mathsf{Con}[\mathsf{Def}]_0$ are separated (in the sense that there are no mixed atoms) in order to test satisfiability of $G_0 \cup \mathsf{Mon}(\Sigma)[G]_0 \cup \mathsf{Con}[\mathsf{Def}]_0$ we need to test entailment of the premises of $\mathsf{Mon}(\Sigma)[G]_0 \cup \mathsf{Con}[\mathsf{Def}]_0$ from G_0; when all premises of some clause are provably true we delete the clause and add its conclusion to G_0. The PTIME assumptions for concept subsumption and for the Ord-Horn fragment ensure that this process terminates in PTIME. □

Example 4 *Consider the special case described in Example 3. Assume that the concepts of sort* num *used in any TBox are of the form* $\uparrow n, \downarrow m$ *and* $[n, m]$.

Consider the TBox $\mathcal{T}$ consisting of the following GCIs:

$$\left\{ \begin{array}{rcl} \exists \mathsf{price}(\downarrow n_1) & \sqsubseteq & \mathsf{affordable}, \\ \exists \mathsf{weight}(\uparrow m_1) \sqcap \mathsf{car} & \sqsubseteq & \mathsf{truck}, \\ \mathsf{has\text{-}weight\text{-}price}(\uparrow m, \downarrow n) & \sqsubseteq & \exists \mathsf{price}(\downarrow n) \sqcap \exists \mathsf{weight}(\uparrow m), \\ \downarrow n & \sqsubseteq & \downarrow n_1, \\ \uparrow m & \sqsubseteq & \uparrow m_1, \\ C & \sqsubseteq & \mathsf{car}, \\ C & \sqsubseteq & \exists\ \mathsf{has\text{-}weight\text{-}price}(\uparrow m, \downarrow n) \end{array} \right\}$$

In order to prove that $C \sqsubseteq_{\mathcal{T}} \mathsf{affordable} \sqcap \mathsf{truck}$ *we proceed as follows. We refute* $\bigwedge_{D \sqsubseteq D' \in \mathcal{T}} \overline{D} \leq \overline{D}' \wedge \overline{C} \not\leq \mathsf{affordable} \wedge \mathsf{truck}$. *We purify the problem introducing definitions for the terms starting with existential restrictions, and express the interval constraints using constraints over* $\mathbb{Q}$ *and obtain the following set of constraints:*

Def	C_{num}	C_{concept}	Mon
$f_{\mathsf{price}}(\downarrow n_1) = c_1$	$n \leq n_1$	$c_1 \leq \mathsf{affordable}$	$n_1 \leq n \rightarrow c_1 \leq c$
$f_{\mathsf{price}}(\downarrow n) = c$	$m \geq m_1$	$d_1 \wedge \mathsf{car} \leq \mathsf{truck}$	$n_1 \geq n \rightarrow c_1 \geq c$
$f_{\mathsf{weight}}(\uparrow m_1) = d_1$		$e \leq c \wedge d$	$m_1 \geq m \rightarrow d_1 \leq d$
$f_{\mathsf{weight}}(\uparrow m) = d$		$C \leq \mathsf{car}$	$m_1 \leq m \rightarrow d_1 \geq d$
$f_{h\text{-}w\text{-}p}(\uparrow m, \downarrow n) = e$		$C \leq e$	
		$C \not\leq \mathsf{affordable} \wedge \mathsf{truck}$	

The task of proving $C \sqsubseteq_{\mathcal{T}} \mathsf{affordable} \sqcap \mathsf{truck}$ *can therefore be reduced to checking if* $C_{\mathsf{num}} \cup C_{\mathsf{concept}} \cup \mathsf{Mon}$ *is satisfiable w.r.t. the combination of SL (sort* concept*) with* $LI(\mathbb{Q})$ *(sort* num*). For this, we note that* C_{num} *entails the premises of the first, second, and fourth monotonicity rules. Thus, we can add* $c \leq c_1$ *and* $d \leq d_1$ *to* C_{concept}. *Thus, we deduce that* $C \leq e \wedge \mathsf{car} \leq (c \wedge d) \wedge \mathsf{car} \leq c_1 \wedge (d_1 \wedge \mathsf{car}) \leq \mathsf{affordable} \wedge \mathsf{truck}$, *which contradicts the last clause in* C_{concept}.

6.2 Extensions of $\mathcal{EL}^+$

Role inclusions for n-ary roles can be also quite useful in applications.

Example 5 *Consider an example similar to Example 3, in we additionally have a ternary role* has-price-place *with arity* (concept, num, concept). *The statement* has-price-place(x, n, p) *means that individual* x *has price* n *at location* p. *The concept*

$$\exists\ \mathsf{has\text{-}price\text{-}place.}(\uparrow n, C) = \{x \mid \exists n' {\geq} n, \exists y \in C \text{ and } \mathsf{has\text{-}price\text{-}place}(x, n', y)\}$$

represents the set of all individuals for which a price below n in some location in C exists. Let is-located *be a binary role. Then*

$$\textsf{has-price-place} \circ \textsf{is-located} \subseteq \textsf{has-price-place}$$

expresses the fact that if the database contains the information that object x has price n in place p, and if p is located in p′ then object x has price n in place p′. (for instance if x has (some) price n in Spain and we know that Spain is located in Europe then x has (some) price n in Europe).

Thus, for roles with arbitrary arity (possibly many-sorted) we can also consider role inclusion constraints of the form $r_1 \circ r_2 \sqsubseteq r$. This means that, for every interpretation $\mathcal{I} = (D, A_1, \dots, A_n)$, if $(x_1, \dots, x_n) \in r_1^{\mathcal{I}}$ and $(x_n, \dots, x_{n+k}) \in r_2^{\mathcal{I}}$ then the tuple $(x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_{n+k}) \in r^{\mathcal{I}}$. The corresponding composition rule at algebraic level is:

$$\begin{aligned}
&f_{\exists r_1}(U_2, \dots, U_{n-1}, f_{\exists r_2}(U_{n+1}, \dots, U_{n+k})) = \\
&\quad = \{y_1 \mid \exists y_i \in U_i, 2 \leq i \leq n-1, \exists y_i \in U_i, n+1 \leq i \leq n+k, \\
&\qquad\qquad (y_n, y_{n+1}, \dots, y_{n+k}) \in r_2^{\mathcal{I}} \text{ and } (y_1, y_2, \dots, y_n) \in r_1^{\mathcal{I}}\} = \\
&\quad = \{y_1 \mid \exists y_i \in U_i, \text{ such that for } 2 \leq i \leq n+k, i \neq n, \\
&\qquad\qquad (y_1, y_2, \dots, y_{n-1}, y_{n+1}, \dots, y_{n+k}) \in r_2^{\mathcal{I}} \circ r_1^{\mathcal{I}}\} \subseteq \\
&\quad \subseteq \{y_1 \mid \exists y_i \in U_i, \text{ such that for } 2 \leq i \leq n+k, i \neq n, \\
&\qquad\qquad (y_1, y_2, \dots, y_{n-1}, y_{n+1}, \dots, y_{n+k}) \in r^{\mathcal{I}}\} = \\
&\quad = f_{\exists r}(U_2, \dots, U_{n-1}, U_{n+1}, \dots, U_{n+k}).
\end{aligned}$$

Theorem 14 *The set of Horn clauses $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$, where the functions in Σ may be n-ary, has the property that every Evans partial model A such that:*

(i) for every $f \in \Sigma$, f_A is a partial function with a finite definition domain;

(ii) for each axiom in RI_a of the form:

$$\forall x_1, \dots, x_{n+k} \\ (f_1(x_1, \dots, x_{n-1}, f_2(x_{n+1}, \dots, x_{n+k})) \leq f(x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_{n+k}))$$

and all $a_1, \dots, a_{n+k} \in A$, if $f_A(a_1, \dots, a_{n-1}, a_{n+1}, \dots, a_{n+k})$ is defined then ${f_2}_A(a_{n+1}, \dots, a_{n+k})$ is defined in A;

(iii) $A \models SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$;

weakly embeds into a total model of $SL \cup \mathsf{Mon}(\Sigma) \cup RI_a$.

Proof: Similar to the proof of Theorem 9. □

7 Conclusions

In this paper we have shown that subsumption problems in $\mathcal{EL}$ can be expressed as uniform word problems in classes of semilattices with monotone operators, and that subsumption problems in $\mathcal{EL}^+$ can be expressed as uniform word problems in classes of semilattices with monotone operators satisfying certain composition laws. This allowed us to obtain, in a uniform way, PTIME decision procedures for $\mathcal{EL}$, $\mathcal{EL}^+$, and extensions thereof. These locality considerations allow us to present a family of PTIME (many-sorted) logics which extend $\mathcal{EL}$ with n-ary roles and/or with numerical domains. Some of these extensions are similar and some are different from other types of extensions studied in the description logic literature such as extensions with n-ary existential quantifiers (cf. e.g. [3]) or with concrete domains [5].

In future work we will use the results presented in this paper and the results in [21, 22, 14] for studying interpolation properties in extensions of $\mathcal{EL}$ and for analyzing possibilities of efficient (modular) reasoning in combinations of ontologies based on extensions of $\mathcal{EL}$. It would be interesting to also investigate which of the complexity (tractability/dichotomy) results for $\mathcal{EL}$ and its extensions (cf. e.g. [16, 15]) are consequences of the properties of their algebraic models.

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